MATH-2400 Sections 13–16

Name: ______________________

Section: ______________________

Instructor: Joe Klobusicky

Exam #3

Please show all work.

Do not use text books, notes, calculators, or other aids.

You may use both sides of one $8\frac{1}{2} \times 11$" crib sheet.

PROBLEM #	POINTS
1 (36 pts)	
2 (18 pts)	
3 (18 pts)	
4 (28 pts)	
Total	

1. (a) Suppose a bar of Albanium has a thermal diffusivity of 5 cm^2/s and a length of $10cm$. For all positive times $t > 0$, at the left end of the bar ($x = 0$), temperature is held constant at 10° Celsius, while the other end ($x = 10$) is held constant at 20° Celsius. Initially, the bar has a uniform temperature of 0° Celsius.

(i) (8 pt) Write down the PDE which describes the temperature $z(x,t)$ of the bar at position x at time $t \geq 0$. Include appropriate boundary and initial conditions.

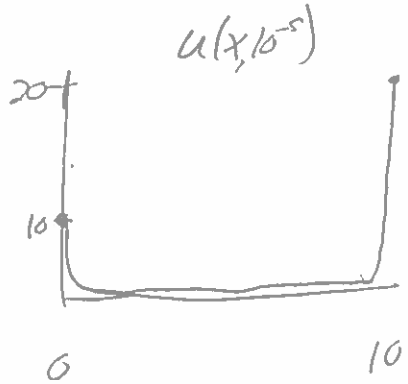
(ii) (6 pt) Sketch an approximation for $z(x,t)$ for $t = .00001$ and $t = 10000$.

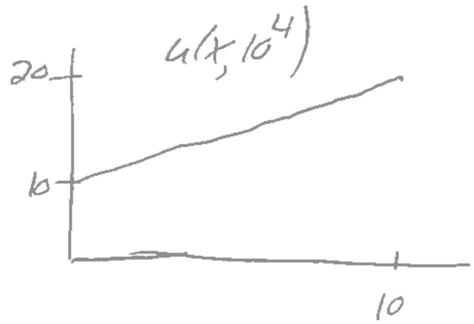
(i) $z_t(x,t) = 5z_{xx}(x,t) \quad 0<x<10, \; t>0$

B.V. $z(0,t) = 10, \; z(10,t) = 20 \quad t>0$

I.C $z(x,0) = 0 \quad 0<x<10$

(ii)





Consider the vertical displacement $u(x,t)$ of a string of length 1 cm with fixed endpoints and a wave speed $a = 3cm/s$. The string has initial displacement of

$$f(x) = 3\sin(\cancel{2x}\ 2\pi x) + 6\sin(\cancel{5x}\ 5\pi x) \quad 0 < x < 1$$

and zero initial velocity.

(i) (12 pt)Assuming a separation of variables $u(x,t) = X(x)T(t)$, derive differential equations for $X(x)$ and $T(t)$, and boundary values for $X(x)$. **Do not solve these equations!**

(ii) (10 pt) The general solution for the problem described above is

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin(n\pi x)\sin(3n\pi t) + d_n \sin(n\pi x)\cos(3n\pi t)$$

Using the initial conditions above, compute the particular solution.

i) Wave equation $u_{tt} = a^2 u_{xx} = 9u_{xx}$ S.O.V. $u(x,t) = X(x)T(t)$

$X\ddot{T} = 9X''T \Rightarrow \frac{X''}{X} = \frac{\ddot{T}}{9T} = -\sigma \Rightarrow X'' + \sigma X = 0,\ \ddot{T} + 9\sigma T = 0.$

B.V. $u(0,t) = X(0)T(t) = 0 \Rightarrow X(0) = 0$

$u(1,t) = X(1)T(t) = 0 \Rightarrow X(1) = 0$

ii) 0 velocity $\Rightarrow u_t(x,0) = \sum_{n=1}^{\infty} c_n \sin(n\pi x)(3n\pi) = 0 \Rightarrow c_n = 0$

$u(x,0) = \sum_{n=1}^{\infty} d_n \sin(n\pi x) = 3\sin(2\pi x) + 6\sin(5\pi x) \Rightarrow d_2 = 3,\ d_5 = 6$

$u(x,t) = 3\sin(2\pi x)\cos(6\pi t) + 6\sin(5\pi x)\cos(15\pi t)$

2. Consider the eigenvalue problem

(1) (2)

$$X'' + \sigma X = 0, \qquad X'(0) = 0, \qquad X(7) = 0.$$

a)(10 pt) Are there positive eigenvalues? If so, what are the corresponding eigenfunctions?
b)(8 pt) Is $\sigma = 0$ an eigenvalue? If so, what is the corresponding eigenfunction?

$\sigma = \lambda^2, \lambda > 0$

a) Ch. eqn $r^2 + \lambda^2 = 0 \Rightarrow r = \pm i\lambda$

$X(x) = C_1 \cos(\lambda x) + C_2 \sin(\lambda x)$

$X'(0) = -\lambda C_1 \sin(\lambda 0) + \lambda C_2 \cos(\lambda 0) = \lambda C_2 = 0 \Rightarrow C_2 = 0$

$X(x) = C_1 \cos(\lambda x)$, $X(7) = C_1 \cos(7\lambda)$. C_1 arb. if $7\lambda = \frac{\pi}{2} + n\pi$, or $\lambda = \frac{\pi(n+\frac{1}{2})}{7}$

So e-vals $\sigma = \lambda^2 = \left(\frac{\pi(n+\frac{1}{2})}{7}\right)^2$. E-fun. $X_n = \cos\left(\frac{\pi(n+\frac{1}{2})}{7} x\right)$ $n \geq 1$.

b) $\sigma = 0 \Rightarrow X'' = 0$, or $X = ax + b$. (1) and (2) state line has zero slope, and hits x-axis. Only $X \equiv 0$ satisfies these conditions, so $\sigma = 0$ is <u>not</u> e-val!

3) (18 pts) In each of the following PDEs for the dependent variable $z = z(x, y)$, determine whether you can separate variables. If not, simply state "No". If so, write down the two differential equations which arise. For all parts, let $z(x,y) = X(x)Y(y)$

(i) $z_{xx} + 2z_y = 0$

$X''Y + 2XY' = 0$

$X''Y = -2XY'$

$\frac{X''}{X} = \frac{-2Y'}{Y} = \sigma \Rightarrow X'' - \sigma X = 0, \quad 2Y' + \sigma Y = 0$

(ii) $z_x + z_y = z$

$X'Y + XY' = XY$

$X'Y = X(Y - Y') \quad \Rightarrow X' - \sigma X = 0$

$\frac{X'}{X} = \frac{Y - Y'}{Y} = \sigma \quad\quad Y' = Y - \sigma Y$

(iii) $yz_{xx} = xz_y + 3$

$yX''Y = xXY' + 3$

$\boxed{\text{No}}$

(iv) $2z_{xx} + z_{yy} = 3x \cdot z(x, y)$

$2X''Y + XY'' = 3xXY$

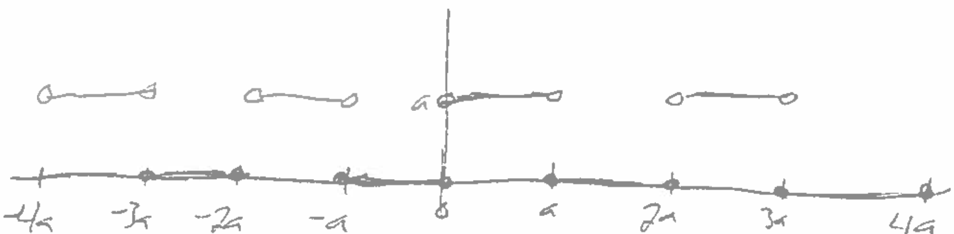
$XY'' = Y(3xX - 2X'') \quad \Rightarrow Y'' - \sigma Y = 0$

$\frac{Y''}{Y} = \frac{3xX - 2X''}{X} = \sigma \quad\quad 2X'' = 3xX - \sigma X$

4) Consider the Fourier series for the **square wave** given by

$$f(x) = \begin{cases} a & 0 < x < a \\ 0 & a \leq x \leq 2a. \end{cases} \qquad f(x+2a) = f(x)$$

(a) (4 pt)Sketch $f(x)$ from $-4a$ to $4a$.



(b) For the Fourier series for $f(x)$ given in the form

$$F(x) = \sum_{n=1}^{\infty} c_n \cos(n\pi x/l) + d_n \sin(n\pi x/l) + \frac{c_0}{2}$$

Compute the following. Note: your answers should not contain any sin or cos terms.

(i)(2pt) l: fund period $= 2a$, so $2l = 2a \Rightarrow l = a$

(ii)(6 pt) c_0:

$$c_0 = \frac{1}{a}\int_{-a}^{a} f(x)\,dx = \frac{1}{a}\int_0^a a\,dx = a$$

(iii) (8 pt) c_n for $n \geq 1$:

$$c_n = \frac{a}{a}\int_0^a \cos\left(\frac{n\pi x}{a}\right)dx = \frac{a}{n\pi}\left[\sin\left(\frac{n\pi x}{a}\right)\Big|_0^a\right]$$

$$= \frac{a}{n\pi}\left[\sin(n\pi) - \sin(0)\right] = \underline{0}$$

(iv) (8 pt) d_n for $n \geq 1$:

$$d_n = \frac{1}{a}\int_0^a a \sin\left(\frac{n\pi x}{a}\right) dx = \frac{a}{n\pi}\left(-\cos\left(\frac{n\pi x}{a}\right)\Big|_0^a\right) = \frac{a}{n\pi}\left(1-\cos(n\pi)\right)$$

$$= \begin{cases} 0 & n \text{ even} \\ \frac{2a}{n\pi} & n \text{ odd.} \end{cases}$$

(c) (3 pt)What is the value of $F(a)$, the Fourier series at $x = a$? (Note: the solution is not needed for this part).

Sol'n converges to midpoint of jump $\frac{F(a^+)+F(a^-)}{2} = \frac{0+a}{2} = \frac{a}{2}.$

Bonus [4pts] Using the Fourier expansion, find an appropriate value of x and a to establish the **Leibniz series**

$$\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = 4(1 - 1/3 + 1/5 - 1/7 + \cdots). \tag{1}$$

Series $f(x) = \frac{a}{2} + \frac{2a}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin\left(\frac{(2n+1)\pi x}{a}\right)$

At $x = \frac{a}{2}$ $f(\frac{a}{2}) = a = \frac{a}{2} + \frac{2a}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin\left(n\pi + \frac{\pi}{2}\right)$ $= (-1)^n$

$$\Rightarrow \frac{\pi}{2} = 2\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} \Rightarrow \pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$$