MATH-2400—Sections 1–4 Mar 29, 2019 NAME: ____________________

Section: ____________________

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Exam 2

Please show all work.

Do not use text books, notes, calculators, or other aids.

You may use one $8\frac{1}{2} \times 11$" crib sheet. Formulas on both sides of the sheet are permitted.

PROBLEM #	POINTS
1 (30 pts)	
2 (36 pts)	
3 (34 pts)	
Total	

1)Consider the system

$$\dot{\mathbf{x}} = \overset{\overset{A}{\parallel}}{\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}} \mathbf{x}.$$

(i) Compute the general solution. (10 pt)

(ii) Sketch trajectories of the phase plane. (10 pt)

(iii)What type of equilibrium occurs at the origin? Classify its stability. (5 pt)

(iv) Compute the particular solution for the initial condition $\mathbf{x}(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$. (5 pt)

(i) $\det(A - Ir) = \begin{vmatrix} 4-r & 2 \\ 2 & 4-r \end{vmatrix} = (4-r)^2 - 4 = 0 \Rightarrow 4 - r = \pm 2$, so $r = 2, 6$

$r_1 = 2$ $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $z_1 + z_2 = 0 \Rightarrow z_2 = -z_1$ $z^{(1)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$r_2 = 6$ $\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $-z_1 + z_2 = 0 \Rightarrow z_2 = z_1$ $z^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

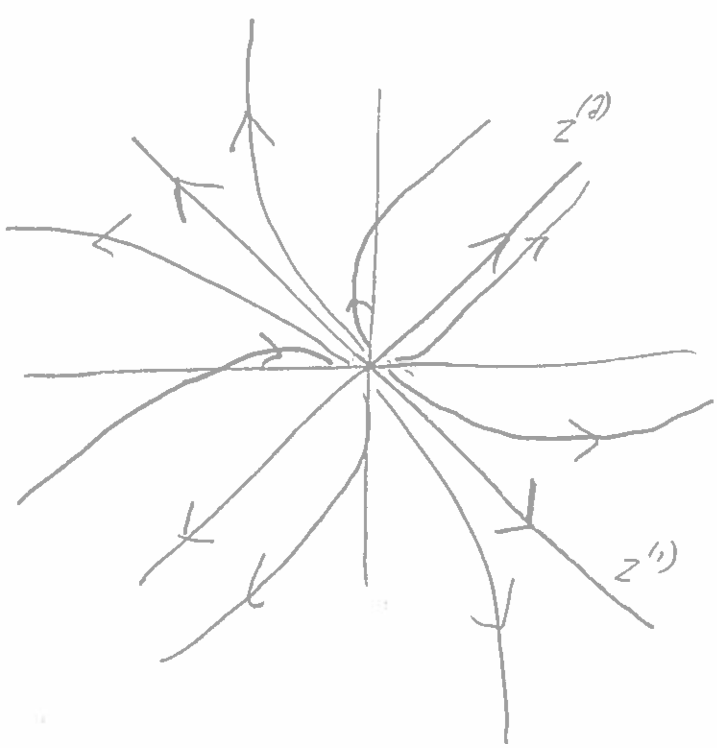
Gen. sol'n

$x(t) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t}$

$x(0) = \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ -c_1 + c_2 \end{bmatrix} \Rightarrow \begin{array}{l} c_1 + c_2 = -1 \\ -c_1 + c_2 = 2 \\ \hline 2c_2 = 1 \\ c_2 = \frac{1}{2}, c_1 = \frac{3}{2} \end{array}$

Part. Sol'n

$x(t) = \frac{3}{2} c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} + \frac{1}{2} c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t}$



Source, unstable equilibrium.

2. Consider a spring modeled by the second order equation

$$\ddot{u}(t) + \gamma \dot{u}(t) + 2u(t) = d\cos(2t).$$

(a) (5 pt) Suppose $d = 0$. What values of $\gamma > 0$, if any, produce a spring which exhibits **damped oscillations**?

(b) (5 pt) Suppose $\gamma = 0$. What values of d, if any, produces a spring system which is **resonant**?

(a) Want $\mathrm{Im}(r) \neq 0$ Ch. Eqn $r^2 + \gamma r + 2 = 0$

$r_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 8}}{2}$ $\mathrm{Im}(r) \neq 0$ when $\gamma^2 - 8 < 0$, or $\gamma < \sqrt{8}$

(b) Ch. eqn. $r^2 + 2 = 0$, $r_{1,2} = \pm\sqrt{2}i$. RHS form $P_n(t)\cos(\beta t)$, $P_n(t) = d$ $(n=0)$ $\beta = 2$.

So $\alpha + i\beta = 2i$ doesn't solve char. eqn $(s=0)$, regardless of what d is!

Thus, spring system is never resonant.

We're still working with the system

$$\ddot{u}(t) + \gamma\dot{u}(t) + 2u(t) = d\cos(2t). \quad (*)$$

(c) (10 pt) Suppose $d = 2$ and $\gamma = 4$. Compute the **stationary solution**.

Stat sol'n = Part. sol'n $u_p(t)$.

Form $u_p(t) = A\cos(2t) + B\sin(2t)$. Plug into $(*)$

$-4A\cos(2t) - 4B\sin(2t) - 8A\sin(2t) + 8B\cos(2t) + 2A\cos(2t) + 2B\sin(2t) = 2\cos(2t)$

Match sin, cos

$-4A + 8B + 2A = 2 \quad (\cos(2t))$
$-4B - 8A + 2B = 0 \quad (\sin(2t))$

$\Rightarrow \begin{matrix} -2A + 8B = 2 \\ -8A - 2B = 0 \end{matrix} \Rightarrow \begin{matrix} B = -4A \\ -A + 4B = 1 \end{matrix} \Rightarrow A = -\frac{1}{17}, B = \frac{4}{17}$

$u_{STAT} = u_p = -\frac{1}{17}\cos(2t) + \frac{4}{17}\sin(2t)$

We're still working with the system

$$\ddot{u}(t) + \gamma \dot{u}(t) + 2u(t) = d\cos(2t). \quad (*)$$

(d) (16 pt) Now suppose $d = 2$ and $\gamma = 0$. Find the general solution for the spring position $u(t)$.

Ch. Eqn. $r^2+2=0$ $r=\pm\sqrt{2}i$

$u_c = C_1\cos(\sqrt{2}t) + C_2\sin(\sqrt{2}t)$

$u_p = A\cos(2t) + B\sin(2t)$. Plug into $(*)$ $-4A\cos(2t) - 4B\sin(2t) + 2A\cos(2t) + 2B\sin(2t) = 2\cos(2t)$

$\Rightarrow -2A = 2, -2B = 0$, so $A = -1, B = 0$

$u_p = -\cos(2t)$

Gen. sol'n $u(t) = (u_c + u_p)(t)$

3) (a) [9 pt] For real numbers α, consider the family of systems $\dot{\mathbf{x}} = \underbrace{\begin{pmatrix} 1 & \alpha \\ 2 & \alpha \end{pmatrix}}_{A} \mathbf{x}$.

For each part below, provide one value of α which produces the type of equilibrium listed. If no such value of α exists, write **DNE**. **Please show all of your work.**

$\det(A - Ir) = \begin{vmatrix} 1-r & \alpha \\ 2 & \alpha - r \end{vmatrix} = (1-r)(\alpha - r) - 2\alpha = r^2 + (-1-\alpha)r - \alpha$ $\quad r_{1,2} = \dfrac{1+\alpha \pm \sqrt{(1+\alpha)^2 + 4\alpha}}{2}$ Call this $g(\alpha)$

(i) Spiral sink

Want $1+\alpha < 0$ and $g(\alpha) < 0$. Test $\alpha = -2$: $1+\alpha = -1$, $g(\alpha) = (1+(-2))^2 + 4(-2) = -7$

(ii) Center

Want $1+\alpha = 0$, $g(\alpha) < 0$. Test $\alpha = -1$: $1+\alpha = 0$, $g(\alpha) = (1-1)^2 + 4(-1) = -4$.

(iii) Saddle

Want $1+\alpha + \sqrt{g(\alpha)} > 0$, and $1+\alpha - \sqrt{g(\alpha)} < 0$. Test $\alpha = 2$: $1+\alpha = 3$, and $\sqrt{g(\alpha)} = \sqrt{3^2 + 8} = \sqrt{17} > 3$

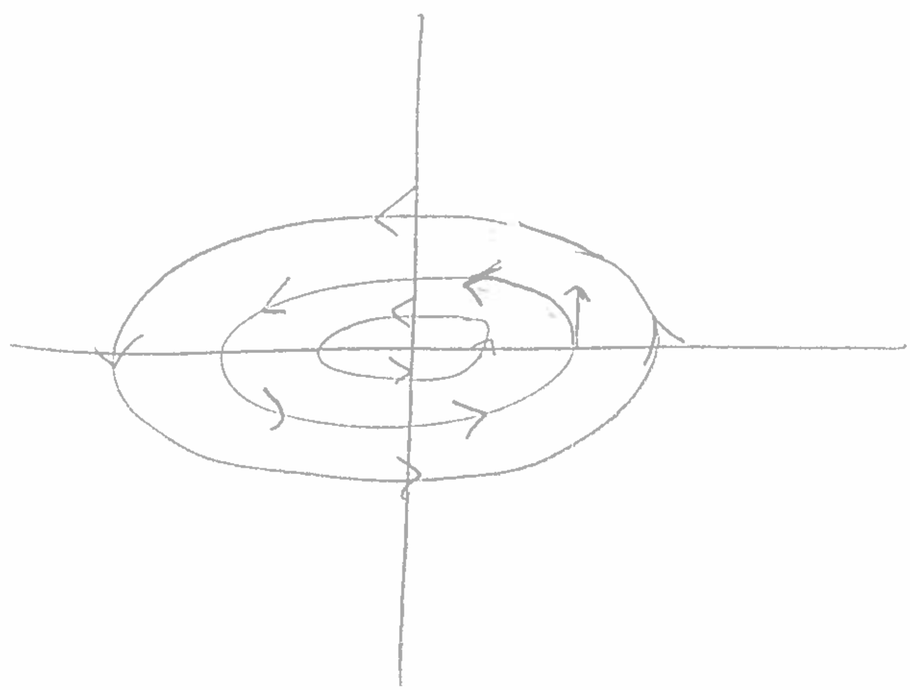
Bonus [3pt]: (iv) Source

Harder! Want $1+\alpha \pm \sqrt{g(\alpha)} > 0$. Test $\alpha = \frac{-1}{8}$. Then $1+\alpha = \frac{7}{8}$, $\sqrt{g(\alpha)} = \sqrt{\left(\frac{7}{8}\right)^2 - \frac{1}{2}} = \sqrt{\frac{17}{64}} = \frac{\sqrt{17}}{8} < \frac{5}{8} < \frac{7}{8}$.

(b) [12 pt] Sketch trajectories for the system of $\dot{\mathbf{x}} = \underbrace{\begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}}_{A} \mathbf{x}$. (Hint: There's no need to compute eigenvectors).

$\det(A - Ir) = \begin{vmatrix} -r & -1 \\ 2 & -r \end{vmatrix} = r^2 + 2 = 0 \Rightarrow r = \pm\sqrt{2}\,i$, center.

Test $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\dot{x} = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$



(c) [8 pt] Consider the complex linear system

$$(2+\beta)x_1 + ix_2 = 0$$
$$ix_1 + \beta x_2 = 0$$

$$\Rightarrow \overset{A}{\begin{bmatrix} 2+\beta & i \\ i & \beta \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Give one value of β which produces a system of equations with more than one solution. For this value of β, give one nonzero solution (x_1, x_2) for this system.

$\det(A) = (2+\beta)(\beta) - i^2 = 0$

$\Rightarrow \beta^2 + 2\beta + 1 = 0 \Rightarrow (\beta+1)^2 = 0$, so $\beta = -1$, then $\begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

so $ix_1 - x_2 = 0$, $x_2 = ix_1$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ i \end{bmatrix}$.

(d) [5 pt] Suppose a 2×2 matrix A has two eigenvalues which are both zero. Does this imply that A has all zero elements? If so, explain why. If not, provide a nonzero matrix with two zero eigenvalues.

Possible sol'ns $\begin{bmatrix} 0 & 0 \\ \alpha & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix}$ for any α.

Can you find sol'n with all nonzero entries?