MATH-2400 Sections 13–16

Name: ______________________________

Section: ______________________________

Instructor: Joe Klobusicky

Exam #3

Please show all work.

Do not use text books, notes, calculators, or other aids.

You may use both sides of one $8\frac{1}{2} \times 11$" crib sheet.

PROBLEM #	POINTS
1 (36 pts)	
2 (18 pts)	
3 (18 pts)	
4 (28 pts)	
Total	

1. (a) Suppose a bar of Albanium has a thermal diffusivity of 5 cm^2/s and a length of $10cm$. For all positive times $t > 0$, at the left end of the bar ($x = 0$), temperature is held constant at 10° Celsius, while the other end ($x = 10$) is held constant at 20° Celsius. Initially, the bar has a uniform temperature of 0° Celsius.

(i) (8 pt) Write down the PDE which describes the temperature $z(x, t)$ in Celsius of the bar at position x and time $t \geq 0$. Include appropriate boundary and initial conditions.

(ii) (6 pt) Sketch an approximation for $z(x, t)$ for $t = .00001$ and $t = 10000$.

(b) Consider the vertical displacement $u(x,t)$ of a string of length 1 cm with fixed endpoints and a wave speed $a = 3cm/s$. The string has initial displacement of

$$u(x,0) = 3\sin(2\pi x) + 6\sin(5\pi x), \quad 0 < x < 1.$$

and zero initial velocity.

(i) (12 pt) Write down the appropriate PDE for $u(x,t)$, and assuming a separation of variables $u(x,t) = X(x)T(t)$, derive differential equations for $X(x)$ and $T(t)$, and boundary values for $X(x)$. **Do not solve these equations!**

(ii) (10 pt) The general solution of the wave equation for the problem above has the form

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin(n\pi x)\cos(3n\pi t) + d_n \sin(n\pi x)\sin(3n\pi t)$$

Using the initial conditions from above, compute the particular solution.

2. Consider the eigenvalue problem

$$X'' + \sigma X = 0, \qquad X'(0) = 0, \qquad X(7) = 0.$$

a)(10 pt) Are there positive eigenvalues? If so, what are the corresponding eigenfunctions?
b)(8 pt) Is $\sigma = 0$ an eigenvalue? If so, what is the corresponding eigenfunction?

3) (18 pts) In each of the following PDEs for the dependent variable $z = z(x, y)$, determine whether you can separate variables. If not, simply state "No". If so, write down the two differential equations which arise.

(a) $z_{xx} + 2z_y = 0$

(b) $z_x + z_y = z$

(c) $yz_{xx} = xz_y + 3$

(d) $2z_{xx} + z_{yy} = 3x \cdot z(x, y)$

4) For some $a > 0$, consider the Fourier series for the **square wave** given by

$$f(x) = \begin{cases} a & 0 < x < a, \\ 0 & a \leq x \leq 2a. \end{cases} \qquad f(x+2a) = f(x).$$

(a) (4 pt) Sketch $f(x)$ for $-4a \leq x \leq 4a$.

(b) For the Fourier series for $f(x)$ given in the form

$$F(x) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\pi x/l) + d_n \sin(n\pi x/l),$$

Compute the following. **Note**: your answers should not contain any sin or cos terms.

(i)(2pt) l :

(ii)(6 pt) c_0 :

(iii) (8 pt) c_n for $n \geq 1$:

(iv) (8 pt) d_n for $n \geq 1$:

(c) (3 pt)What is the value of $F(a)$, the Fourier series at $x = a$? (Note: the solution is not needed for this part).

Bonus [4pts] By substituting an appropriate value of x and a into the Fourier series $F(x)$, derive the **Leibniz series**

$$\pi = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 4(1 - 1/3 + 1/5 - 1/7 + \cdots).$$