MATH-2400 Sections 17–20

Name: ______________________________

Section: ______________________________

Instructor: Joe Klobusicky

Exam #3

Please show all work.

Do not use text books, notes, calculators, or other aids.

You may use both sides of one $8\frac{1}{2} \times 11$" crib sheet.

PROBLEM #	POINTS
1 (40 pts)	
2 (18 pts)	
3 (26 pts)	
4 (16 pts)	
Total	

1. Let $u(x,t)$ describe temperature in a bar of unobtanium having thermal diffusivity of 1 cm^2/s and length $10cm$. Suppose both sides at all positive times maintain a temperature of 0 degrees Celsius. In this case, $u(x,t)$ satisfies the heat equation

$$u_t = u_{xx},$$

with boundary conditions

$$u(0,t) = 0, \qquad u(10,t) = 0.$$

The bar has an initial temperature profile of

$$u(x,0) = \begin{cases} 10 & 0 < x < 5, \\ 0 & 5 \le x < 10. \end{cases} = f(x)$$

a) [16 pts.] Assuming a separation of variables $u(x,t) = X(x)T(t)$, derive ODEs for $X(x)$ and $T(t)$, and boundary values for $X(x)$. (You do not need to solve these ODEs).

b) [16 pts.] The general solution for the problem described above is

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-(n\pi/10)^2 t} \sin(n\pi x/10). \quad (*)$$

Compute the particular solution corresponding to the initial conditions given above.

c)[8 pts.] Sketch approximate solutions for $u(x, .000001)$ and $u(x, 10^6)$.

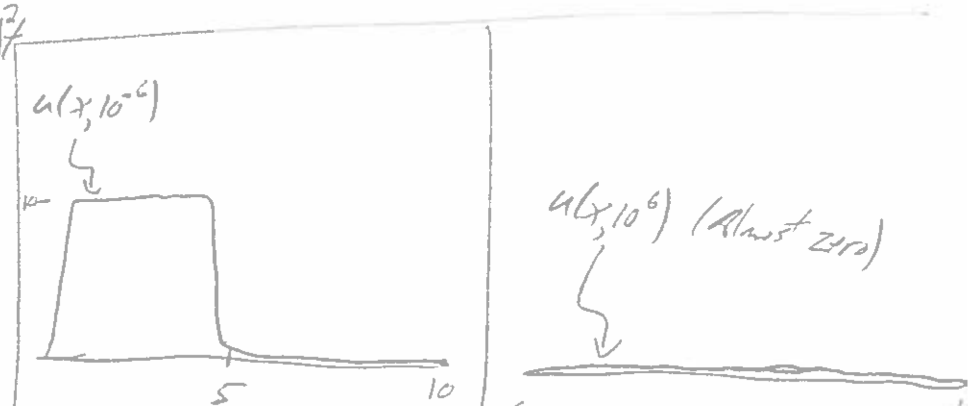
a) $u(x,t) = X(x)T(t) \Rightarrow XT' = X''T \Rightarrow \frac{X''}{X} = \frac{T'}{T} = -\sigma$ for σ constant, thus $X'' + \sigma X = 0,\ T' + \sigma T = 0$

B.C.s $u(0,t) = X(0)T(t) = 0$, $u(10,t) = X(10)T(t) = 0$ $\Rightarrow X(0) = X(10) = 0$.

b) $u(x,0) = f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{10}\right)$ from (*), This is a sin series, so

$$c_n = \frac{2}{10}\int_0^{10} f(x)\sin\left(\frac{n\pi x}{10}\right)dx = \frac{1}{5}\int_0^5 10\sin\left(\frac{n\pi x}{10}\right)dx = \frac{-2\cdot 10}{n\pi}\left(\cos\left(\frac{n\pi x}{10}\right)\Big|_0^5\right) = \frac{20}{n\pi}\left(1-\cos\left(\frac{n\pi}{2}\right)\right)$$

$$u(x,t) = \frac{20}{\pi}\sum_{n=1}^{\infty}\left(\frac{1-\cos\left(\frac{n\pi}{2}\right)}{n}\right)\sin\left(\frac{n\pi x}{15}\right)e^{-\left(\frac{n\pi}{15}\right)^2 t}$$



2. Consider the eigenvalue problem

$$X'' + \sigma X = 0, \qquad \overset{(*)}{X(-1) = 0}, \qquad \overset{(**)}{X'(0) = 0}.$$

a)[10 pt] Are there positive eigenvalues? If so, what are the corresponding eigenfunctions?
b)[8 pt] Is $\sigma = 0$ an eigenvalue? If so, what is the corresponding eigenfunction?

a) Let $\sigma = \mu^2$. Then $X'' + \mu^2 X = 0 \Rightarrow X(x) = C_1 \sin(\mu x) + C_2 \cos(\mu x)$

$X'(x) = \mu C_1 \cos(\mu x) - \mu C_2 \sin(\mu x)$, $X'(0) = \mu C_1 = 0 \Rightarrow C_1 = 0$, so

$X(x) = C_2 \cos(\mu x)$, Then $X(-1) = C_2 \cos(-\mu \cdot) = C_2 \cos(\mu\ ) = 0$.

If $C_2 \neq 0$, $\mu = (n+\frac{1}{2})\pi$ for $n = 1, 2, \dots$. Then $((n+\frac{1}{2})\pi)^2 = \sigma_n$ are e-vals, w/ corr.

e-functions $\cos((n+\frac{1}{2})\pi x) = X_n$.

b) If $\sigma = 0$, $X'' = 0$, so $X = Ax + b$, a line. A line which has slope 0 (from (**)) and passes through the x axis (from (*)) must be identically 0, so $X(x) = 0$, meaning that 0 is not an eigenvalue.

3. [20 pts.] (a)Compute the cosine expansion for the function $f(x) = \sin(x)$ defined on $0 < x < \pi$.

Hint: $2\sin(a)\cos(b) = \sin(a+b) + \sin(a-b)$. $(*)$

(b) [6 pt] Sketch what this expansion will look like in the region $-2\pi < x < 2\pi$.

Let $F(x)$ be cosine rep for $f(x)$ on $-\pi < x < \pi$.

Here, $\ell = \pi$, and $F(x) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{\pi}\right) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(nx)$

$$c_n = \frac{2}{\pi}\int_0^{\pi} \cos(nx)\sin(x)\,dx \underset{(*)}{=} \frac{1}{\pi}\int_0^{\pi} \sin((n+1)x) + \sin((1-n)x)\,dx = \frac{1}{\pi}\left(\frac{1}{n+1}\left(1-\cos((n+1)\pi)\right) + \frac{1}{1-n}\left(1-\cos((1-n)\pi)\right)\right)$$

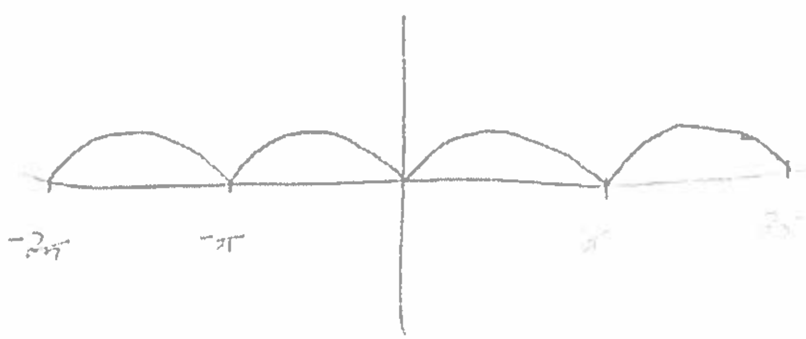
$1-\cos((n+1)\pi) = \begin{cases} 2 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$ so $c_n = 0$, if n odd.

$1-\cos((1-n)\pi) = \begin{cases} 2 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$

If n even, $c_n = \frac{2}{\pi}\left(\frac{1}{n+1} - \frac{1}{n-1}\right) = -\frac{4}{\pi(n^2-1)}$. Thus $F(x) = \frac{2}{\pi} - \sum_{n=1}^{\infty} \frac{4\cos(2nx)}{\pi((2n)^2-1)}$

$\left(c_0 = -\frac{4}{\pi(0-1)} = \frac{4}{\pi}\right)$

b) Since cos expansion is even,



4. Consider the following PDE used in **renewal theory** describing waiting times $u(x,t)$ for customers in a line:

$$u_t(x,t) - u_x(x,t) = u(0,t)p(x). \quad x,t \geq 0. \tag{1}$$

Here $p(x)$ is some positive function defined on $[0,\infty)$.

(a)[16 pt] Assuming a solution of the form $u(x,t) = X(x)T(t)$, find ordinary differential equations satisfied by $X(x)$ and $T(t)$. **Note**: do not attempt to solve these equations!
Bonus [3 pt] If $p(x)$ satisfies $\int_0^\infty p(x)dx = 1$ and $u(x,t) \to 0$ as $x \to \infty$, show that the total number $N(t) = \int_0^\infty u(x,t)dx$ is conserved, meaning that $N(t) = N(0)$ for all $t > 0$.

a) $u(x,t) = X(x)T(t)$

$\Rightarrow X(x)T'(t) - X'(x)T(t) = X(0)T(t)p(x)$. Divide by $X(x)\,T(t)$

$$\frac{T'(t)}{T(t)} - \frac{X'(x)}{X(x)} = \frac{X(0)p(x)}{X(x)} \Rightarrow \frac{T'(t)}{T(t)} = \frac{X(0)p(x)+X'(x)}{X(x)} = \sigma, \ \sigma \text{ some constant}$$

Thus $T'(t) - \sigma T(t) = 0$

$X'(x) - \sigma X(x) + X(0)p(x) = 0$

[Bonus] Integrate (1) wrt x

$$\int_0^\infty u_t(x,t)dx - \int_0^\infty u_x(x,t)dx = \int_0^\infty u(0,t)p(x)dx$$

$$\Rightarrow N'(t) - \left(u(x,t)\Big|_0^\infty\right) = u(0,t)\int_0^\infty p(x)dx = u(0,t)$$

$N'(t) + u(0,t) = u(0,t) \Rightarrow N'(t) = 0$, so $N(t)$ is constant!

Thus, $N(t) = N(0)$.