MATH-2400—Sections 13–16 Nov 5, 2018

NAME: ______________________________

Section: ______________________________

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Exam #2

Please show all work.

Do not use text books, notes, calculators, or other aids.

You may use one $8\frac{1}{2} \times 11$" crib sheet. Formulas on both sides of the sheet are permitted.

PROBLEM #	POINTS
1 (40 pts)	
2 (35 pts)	
3 (25 pts)	
Total	

1.

(a)A spring system with a mass of $m > 0$ is modeled by

$$m\ddot{u}(t) + 5\dot{u}(t) + 2u(t) = 0, \quad u(0) = 0, \quad \dot{u}(0) = 1.$$

where units of time are measured in seconds, distance is measure in meters, and mass in kilograms.

(i) (5 pt) How large is the force of resistance on the mass when it has a velocity of 3 meters per second? (Note: the units for the linear damping constant γ are kg/s.)

(ii) (5 pt)For what value of m is the system critically damped? (**Note**: Show all work)

(iii) (15 pt) For $m = 2$ kg, compute the particular solution for $u(t)$.

(i) $F_{damp} = \dot{u} \cdot \gamma = \left(3 \frac{m}{s}\right)\left(5 \frac{Kg}{s}\right) = 15\, Kg\, m/s^2$.

(ii) Ch. eqn $mr^2 + 5r + 2 = 0$ $r_{1,2} = \dfrac{-5 \pm \sqrt{25 - 8m}}{2m}$.

Repeated root if $25 - 8m = 0 \Rightarrow m = \frac{25}{8}$ Kg.

(iii) From (ii), $r_{1,2} = \dfrac{-5 \pm \sqrt{9}}{4} \Rightarrow r = -2, -\frac{1}{2}$.

$u(t) = C_1 e^{-2t} + C_2 e^{-\frac{t}{2}}$. $0 = u(0) = C_1 + C_2$ (*)

$1 = \dot{u}(0) = -2C_1 - \frac{C_2}{2}$ (**)

(*) + 2(**) $\Rightarrow 2 = -3C_1$

$C_1 = -\frac{2}{3}, C_2 = \frac{2}{3}$

$u(t) = -\frac{2}{3} e^{-2t} + \frac{2}{3} e^{-\frac{t}{2}}$

(b)For some spring constant $k > 0$, another spring system is given by

$$(*) \qquad \ddot{u}(t) + ku(t) = \sin(2t), \quad u(0) = 0, \dot{u}(0) = 0.$$

(i) (5 pt) What value of $k > 0$ would make the system resonant?

(ii)(10 pt) Compute the general solution of $u(t)$ in the case of $k = 3$ kg/s^2.

(i) Ch. eqn $r^2 + k = 0 \Rightarrow r = \pm i\sqrt{k}$

$\alpha + i\beta = 2i \Rightarrow \sqrt{k} = 2 \Rightarrow k = 4$ kg/s^2

(ii) Comp sol'n. $r = \pm i\sqrt{3}$ $u_c = C_1 \sin(\sqrt{3}t) + C_2 \cos(\sqrt{3}t)$

Part sol'n $u_p(t) = A\sin(2t) + B\cos(2t)$. Plug u_p into $(*)$

$\ddot{u}_p + 3u_p = -4A\sin(2t) - 4B\cos(2t) + 3A\sin(2t) + 3B\cos(2t) = \sin(2t)$

Matching $\sin(2t)$ and $\cos(2t)$:

$-A\sin(2t) = \sin(2t) \Rightarrow A = -1$

$-B\cos(2t) = 0 \Rightarrow B = 0$

$u_p = -\sin(2t)$

Gen sol'n $u(t) = u_c(t) + u_p(t)$

2. (a) For a some real number α, consider the system $\dot{\mathbf{x}} = \underbrace{\begin{pmatrix} 1 & 1 \\ -2 & \alpha \end{pmatrix}}_{=A} \mathbf{x}$.

For each of the following cases, provide **one value** of α which makes the equilibrium (i) a center (3 pt) ,(ii) a spiral sink (3 pt) , and (iii) a source. (3 pt)

What is the stability type for each of these equilibria? (3 pt)

(b) Determine whether the orbits for the system

$$\dot{\mathbf{x}} = \begin{pmatrix} 3 & 6 \\ -4 & -3 \end{pmatrix} \mathbf{x}$$

travel clockwise or counterclockwise. (5 pt)

Reminder: for all parts, you must provide a justification for your answer. A correct answer with no explanation is worth zero points.

$$\det(A - Ir) = \begin{vmatrix} 1-r & 1 \\ -2 & \alpha - r \end{vmatrix} = (1-r)(\alpha - r) + 2 = r^2 + (-\alpha - 1)r + 2 + \alpha = 0$$

$$r = \frac{1+\alpha \pm \sqrt{(1+\alpha)^2 - 4(2+\alpha)}}{2}$$

(i) $\alpha = -1$, since $(1+\alpha)^2 - 4(2+\alpha) < 0$ and $1+\alpha = 0$

$\Rightarrow Re(r) = 0$ and $Im(r) \neq 0$

(ii) Want $1+\alpha < 0$ and $(1+\alpha)^2 - 4(2+\alpha) < 0$

Test $\alpha = -\frac{3}{2}$. Then $1+\alpha = -\frac{1}{2}$ and $(1+\alpha)^2 - 4(2+\alpha)$

$= \frac{1}{4} - 2 < 0$ ✓✓

(iii) Want 2 positive roots, choose very large α, say

$\alpha = 10^6$. Then $1+\alpha > 0$, and $(1+\alpha)^2 - 4(2+\alpha) > 0$

and $(1+\alpha)^2 - 4(2+\alpha) > (1+\alpha)^2$, so $r > 0$.

(c) Consider the system

$$\dot{\mathbf{x}} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \mathbf{x}.$$

(i) Compute the general solution. (10 pt)

(ii) Sketch trajectories of the phase plane. (5 pt)

(iii) Compute the particular solution for the initial condition $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. (3 pt)

$\det(A - Ir) = \begin{vmatrix} 1-r & 2 \\ 2 & 1-r \end{vmatrix} = (1-r)^2 - 4 = 0 \Rightarrow (1-r)^2 = 4$

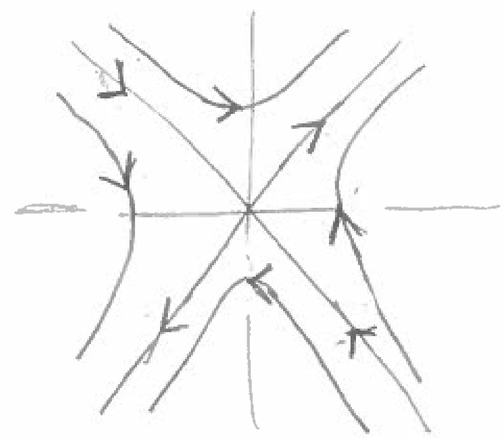
$1-r = \pm 2 \Rightarrow r = -1, 3$

$r_1 = -1 \quad \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = 0 \Rightarrow 2z_1 + 2z_2 = 0 \Rightarrow z_2 = -z_1 \quad z^{(1)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$r_2 = 3 \quad \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = 0 \Rightarrow -2z_1 + 2z_2 = 0 \Rightarrow z_2 = z_1 \quad z^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Gen sol'n

$x(t) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}$



Part sol'n $\begin{bmatrix} 1 \\ 2 \end{bmatrix} = x(0) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow 1 = c_1 + c_2$

$2 = -c_1 + c_2$

$3 = 2c_2 \Rightarrow c_2 = \frac{3}{2}, \quad c_1 = -\frac{1}{2}$

$x(t) = -\frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + \frac{3}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}$

3. (a) (15 pt) Give the **general form** of the solution for the second order equation

$$y''(t) + y(t) = 7te^{-3t} + t^3\cos(t).$$

(**Note**: you do not need to determine values for undetermined or arbitrary coefficients)
(b) (10 pt) Find the general solution for the second order equation

$$2(x-3)^2y''(x) + 3(x-3)y'(x) + y(x) = 0, \quad x \neq 3.$$

(a) y_c, Ch eqn $r^2+1=0 \Rightarrow r=\pm i$ $y_c = C_1\cos(t) + C_2\sin(t)$

y_{p_1} RHS for $P_1(t)e^{\alpha t}$ $P_1(t)=7t$ $\alpha=-3$, $\alpha+i\beta=-3$ $(s=0)$

$y_{p_1} = (At+B)e^{-3t}$

y_{p_2} RHS $P_3(t)\cos(\beta t)$ $P_3(t)=t^3$, $\beta=1$ $\alpha+i\beta=i$ $(s=1)$

$y_{p_2} = t\left[(Ct^3+Dt^2+Et+F)\cos(t) + (Ht^3+It^2+Kt+L)\sin(t)\right]$

$y = y_c + y_{p_1} + y_{p_2}$

(b) Ind eqn. $2r(r-1)+3r+1 = 2r^2+r+1=0 \Rightarrow r = \frac{-1\pm\sqrt{1-8}}{4} = -\frac{1}{4} \pm \frac{\sqrt{7}}{4}i$

$y(x) = C_1|x-3|^{-\frac{1}{4}}\cos\left(\frac{\sqrt{7}}{4}\log(|x-3|)\right) + C_2|x-3|^{-\frac{1}{4}}\sin\left(\frac{\sqrt{7}}{4}\log(|x-3|)\right)$