MATH-2400 NAME:__________________________

Instructor: Joe Klobusicky Friday, February 22, 2019

Exam 1

Please answer all questions, showing your work in detail and giving reasons for your conclusions.

You may use **both sides of a** (two-sided) $8\frac{1}{2} \times 11$ sheet
in your own handwriting, but no other notes, books, computers, calculators, cell phones, or other references or communication tools are permitted.

Please circle your section (by recitation):

1: Tuesday 8 am **2: Friday 8 am**

3: Tuesday 9 am **4: Friday 9 am**

Problem	Points
1/26 pts.	
2/16 pts.	
3/28 pts.	
4/30 pts.	
TOTAL	

1. Consider the following four differential equations given by

(A) $\dot{N}(t) + N(t) = e^{-t}\sin(t)$ (C) $\dot{N}(t) = e^{N(t)}$

(B) $t^2\ddot{N}(t) + 3t\dot{N}(t) + N(t) = 0$ (D) $\dot{N}(t) - N(t)^2 = 0$

Circle the letters corresponding to which equations are

(a) [4 pt] **Linear**:

A B C D

(b) [4 pt] **Autonomous:** (meaning that it is possible to draw a phase line for the equation)

A B C D

(c) [4 pt] For large N, which grow faster: solutions of (C) or (D)? In one sentence, why?

(c) Comparing rates of growth, $e^N > N^2$ for large N.

We're still dealing with the four differential equations

(A) $\dot{N}(t) + N(t) = e^{-t}\sin(t)$ (C) $\dot{N}(t) = e^{N(t)}$

(B) $t^2\ddot{N}(t) + 3t\dot{N}(t) + N(t) = 0$ (D) $\dot{N}(t) - N(t)^2 = 0$

(d) [6 pt] Do any of the models (A), (C), or (D) have any equilibrium solutions (meaning $N(t) = C$ for some constant C for all t)? What are they?

Only (D) does. By setting $\dot{N}=0$,

(A) $N(t) = e^{-t}\sin(t)$ (not constant)

(C) $0 = e^{N(t)}$ (not possible)

(D) $-N^2 = 0 \Rightarrow \underline{N(t) \equiv 0}$ for all t.

(e) [8 pt] Find the general solution for equation (B) for all $t \neq 0$.

<u>Euler Eqn.</u> <u>Indicial Eqn</u>: $r(r-1)+3r+1 = r^2+2r+1 = (r+1)^2$

<u>Roots</u> $r_{1,2} = -1$

<u>Sol'n</u>: $N(t) = C_1 |t|^{-1} + \frac{C_2 \log(|t|)}{|t|} = \frac{C_1 + C_2\log(|t|)}{|t|}$

2. Bactolabs Inc. grows a culture in a petri dish with a population of $N(t)$ bacteria after t days. The bacteria grown at a rate proportional to its population, with a growth constant r. The lab is unsanitary, and incompetent lab assistants consistently cough, causing an approximate continuous addition of k bacteria per day to be added to the dish.

(a) [2 pts] Should r be positive or negative?

(b)[4 pts] Write down a differential equation describing the population $N(t)$ at time $t \geq 0$.

(c) [10 pts] The lab distributes face masks, eliminating outside bacteria (meaning $k = 0$). Suppose the dish initially has N_0 bacteria. At what time should the population triple? Your answer should only contain r and/or N_0.

(a) Positive.

(b) $\dot{N}(t) = rN(t) + K$

(c) When $K=0$ $\dot{N} = rN \Rightarrow \frac{dN}{N} = r\,dt \Rightarrow \log(N) = rt + \log C$

$N = Ce^{rt}$

Init conds $N(0) = N_0 = Ce^{r(0)} = C$, so $N(t) = N_0 e^{rt}$

Let τ = time to triple Then

$3N_0 = N(\tau) = N_0 e^{r\tau} \Rightarrow 3 = e^{r\tau} \Rightarrow \log(3) = r\tau$, so $\tau = \frac{\log(3)}{r}$.

3. Bactolabs Inc. has also discovered a new strain of bacteria. A group of interns model growth in a Petri dish by the differential equation

$$\dot{N}(t) = N(t)^2 - 13N(t) + 30,$$

where N measures bacteria in the thousands and time t is measured in hours.

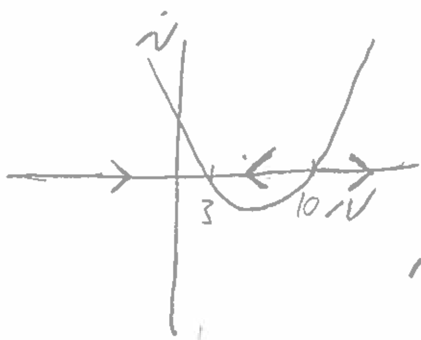
(a) [12 pts.] Draw the phase line and plot the N vs. $\dot{N}$ graph. What is the value and stability type for each of the equilibria?

(b) [12 pts.] Sketch representative integral curves on the t, N plane for $t > 0$.

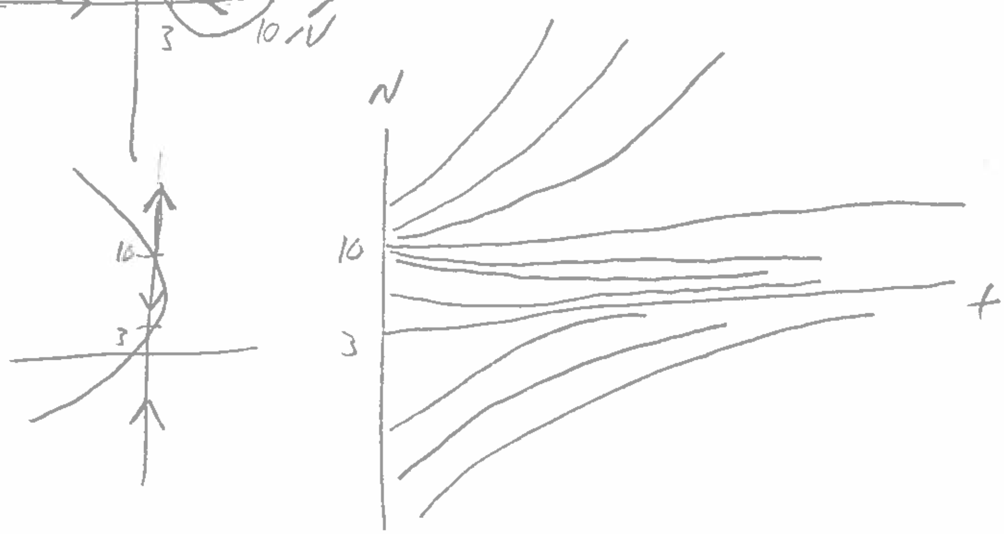
(c) [4 pt] Consider the initial population $N(0) = 0$, and explain in a sentence why the differential equation above is a bad model.

(a) $\dot{N} = (N-10)(N-3)$

Eq. at $N=3$ (stable)
$N=10$ (unstable)



(b)



c) Init pop'n of 0 should remain at 0, but model states $\dot{N}(0) = 30 > 0$!

4. (a) [15 pts.] Compute the solution of the initial value problem

$$4y'' - y' = x \qquad y(0) = 0, \quad y'(0) = 1. \qquad (*)$$

Comp. sol'n

$4y_c'' - y_c' = 0$ Char. eqn $4r^2 - r = 0 \Rightarrow r = 0, r = \frac{1}{4}$

$r(4r-1) = 0$

So $y_c = C_1 + C_2 e^{\frac{x}{4}}$.

Part sol'n RHS $f = P_n(x) = x$ $n=1$. $\alpha + i\beta = 0$ solves char. eqn, so $s = 1$.

Form is $y_p(x) = x[Ax+B] = Ax^2 + Bx$. Plug into $(*)$, with $y_p'' = 2A$, $y_p' = 2Ax + B$

$4y_p'' - y_p' = 8A - 2Ax - B = x \Rightarrow -2A = 1 \quad (x) \Rightarrow A = -\frac{1}{2}$

$8A - B = 0 \quad (1) \qquad B = -4$

So $y_p = -\frac{x^2}{2} - 4x$

Gen sol'n $y(x) = C_1 + C_2 e^{\frac{x}{4}} - \frac{x^2}{2} - 4x$

$0 = y(0) = C_1 + C_2$

$1 = y'(0) = \frac{C_2}{4} - 4$

$\Rightarrow C_2 = 20$, $C_1 = -20$

$\Rightarrow y(x) = -20 + 20e^{\frac{x}{4}} - \frac{x^2}{2} - 4x$

4. (b) [15 pts.] Compute the **general form** for the solution of

$$y'' - 2y' + 5y = e^x(\sin(2x) + 3x),$$

Note: for this problem, you DO NOT need to compute any undetermined coefficients

Comp. sol $r^2 - 2r + 5 = 0$ $r = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$

$y_c = C_1 e^x \cos(2x) + C_2 e^x \sin(2x)$

y_{p_1}: RHS $e^x \sin(2x)$ $\alpha + i\beta = 1 + 2i$ solves char. eqn so $s=1$.

For $y_{p_1} = x[Ae^x \sin(2x) + Be^x \cos(2x)]$

y_{p_2} RHS $P_n(x)e^x$ $P_n(x) = 3x$ $(n=1)$ $\alpha + i\beta = 1$ does not solve ch. eqn $(s=0)$

$y_{p_2} = (Cx + D)e^x$

Then gen sol'n is $y(x) = y_c + y_{p_1} + y_{p_2}$.