MATH-2400 NAME: _____

Instructor: Joe Klobusicky Thursday, October 4, 2018

Exam 1

Please answer all questions, showing your work in detail and giving reasons for your conclusions.

You may use **both sides of a** (two-sided) $8\frac{1}{2} \times 11$ sheet
in your own handwriting, but no other notes, books, computers, calculators, cell phones, or other references or communication tools are permitted.

Please circle your section: **13** **14** **15** **16**

Problem	Points
1/16 pts.	
2/30 pts.	
3/30 pts.	
4/24 pts.	
TOTAL	

1. (a) [16 pts.] Find the solution of the initial-value problem

$$ty(t)' - 2y(t) - t = 0, \qquad y(1) = 1.$$

Sketch a solution for the equation.

Rewrite as $y'(t) - \frac{2}{t}y = 1$

$$\mu(t) = e^{-\int \frac{2}{t}\,dt} = e^{-2\log(t)} = e^{\log(t^{-2})} = \frac{1}{t^2}$$

$$[\mu(t)y(t)]' = \mu(t)\cdot 1$$

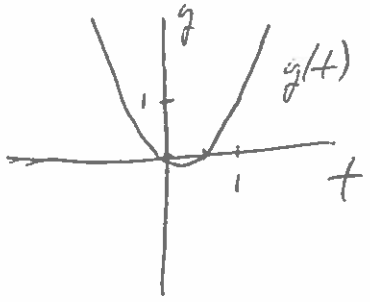
$$\left[\frac{1}{t^2}y(t)\right]' = \frac{1}{t^2}$$

$$\frac{1}{t^2}y(t) = -\frac{1}{t} + C$$

$$y(t) = -t + Ct^2$$

I.C. $1 = y(1) = -1 + C$, so $y(t) = -t + 2t^2 = t(2t-1)$

$\Rightarrow C = 2$



2. The Troy Power Company has decided to switch to nuclear power to run its facility. Specifically, it relies on a supply of the radioactive (and fictional) material of Albanium, with a decay rate of 3/year. Moreover, the manager of the company has decided to continuously add Albanium at a constant rate of 3kg/year. Suppose the company at time $t = 0$ begins with 2 kg of Albanium.

(a) [6 pts.] Write down an initial value problem (differential equation + initial conditions) for the amount $Q(t)$ of Albanium at time $t \geq 0$.

(b) [12 pts.] Solve the initial value problem.

(c) [6 pts.] If the amount of Albanium reaches 5 kg, the facility goes into panic mode. When, if ever, does panic mode happen?

(d) [6 pts.] The manager chose to add 3kg of Albanium a year because he thought it would balance out the loss due to the decay rate of 3/year. This is incorrect reasoning, however. What value should the manager choose in order for the amount of Albanium to remain constant at 2 kg for all times?

d) Want $Q' = 0$. Let constant rate be K

Then $Q' = -3Q + K$

$\Rightarrow 0 = -3Q(0) + K = -6 + K$

$K = 6 \frac{kg}{yr.}$

a) $\frac{dQ}{dt} = -3Q + 3 \quad Q(0) = 2$

b) $Q' + 3Q = 3 \quad \mu(t) = e^{\int 3dt} = e^{3t}$

$(e^{3t}Q)' = 3e^{3t}$

$e^{3t}Q = e^{3t} + C$

$Q(t) = 1 + Ce^{-3t}$

I.C. $2 = Q(0) = 1 + C \Rightarrow C = 1$

$Q(t) = 1 + e^{-3t}$

c) $Q(t) = 1 + e^{-3t} < 5$, so panic never occurs.

3. Consider an autonomous first order differential equation, given by

$$\frac{dM}{dt} = M^2(M-2)^2.$$

(a) [12 pts.] Draw the phase line and plot the M' vs. M graph. What is the value and stability type for each of the equilibria?

(b) [12 pts.] Sketch representative integral curves on the t, M plane.

(c) [6 pts.] What happens to $M(t)$ as $t \to \infty$ if

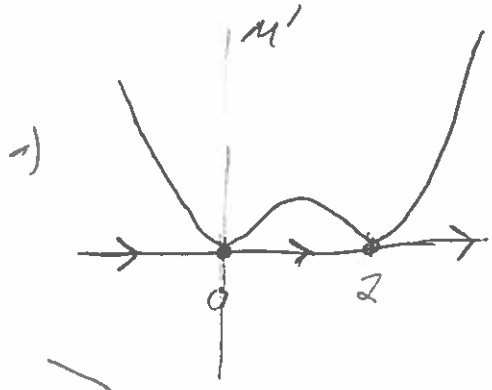
(i) $M(0) = -2$,

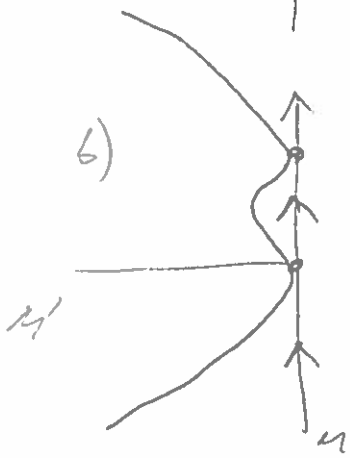
(ii) $M(0) = 1$,

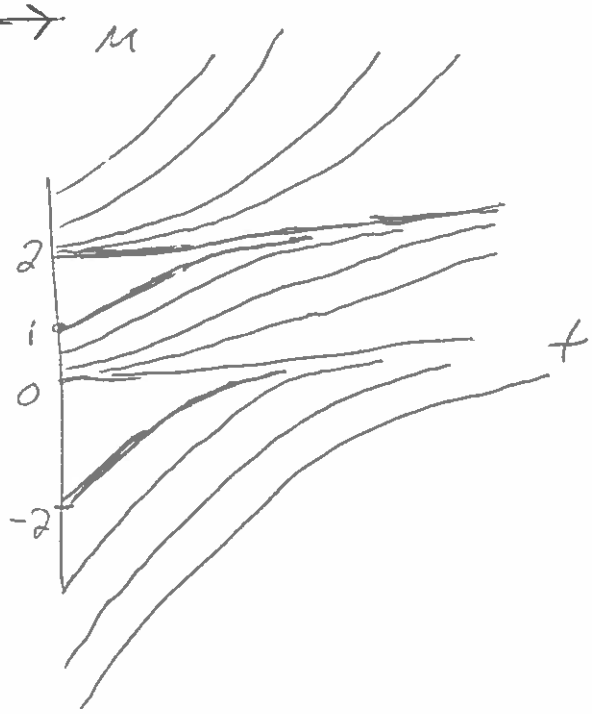
(iii) $M(0) = 2$

Equil at $M=0$ semi-stable

$M=2$ semi-stable.







As $t \to \infty$

c) (i) $M(t) \to 0$

(ii) $M(t) \to 2$

(iii) $M(t) = 2$ for all $t \geq 0$.

4. (a) [12 pts.] Compute the solution of the initial value problem

$$4y'' - 4y' + y = 0 \qquad y(0) = 0, \quad y'(0) = 1.$$

Ch. Eqn $4r^2 - 4r + 1 = 0$

$(2r-1)^2 = 0$ (or $r_{1,2} = \frac{4 \pm \sqrt{16-16}}{8} = \frac{1}{2}$

$r = \frac{1}{2}$ Repeated.

$y(t) = C_1 e^{\frac{t}{2}} + C_2 t e^{\frac{t}{2}}$

$y(0) = C_1 = 0$

$y(t) = C_2 t e^{\frac{t}{2}}$, $y'(t) = C_2 e^{\frac{t}{2}} + \frac{1}{2} C_2 t e^{\frac{t}{2}}$

$1 = y'(0) = C_2$

$\Rightarrow y(t) = t e^{\frac{t}{2}}$

4. (b) [12 pts.] Compute the solution of the initial value problem

$$y'' - 4y' + 20y = 0, \qquad y(0) = 0, \quad y'(0) = -1.$$

Ch. Eqn. $r^2 - 4r + 20$

$$r_{1,2} = \frac{4 \pm \sqrt{16-80}}{2} = 2 \pm 4i$$

λ μ

$$y(t) = C_1 e^{2t}\cos(4t) + C_2 e^{2t}\sin(4t)$$

$$y(0) = C_1 = 0 \Rightarrow y(t) = C_2 e^{2t}\sin(4t)$$

$$y'(t) = 2C_2 e^{2t}\sin(4t) + 4C_2 e^{2t}\cos(4t)$$

$$y'(0) = 4C_2 = -1 \Rightarrow C_2 = -\frac{1}{4}$$

$$y(t) = -\frac{1}{4} e^{2t}\sin(4t)$$