

Suggested Homework 6

In each of the following problems, find the eigenvalues and eigenfunctions of the given boundary-value problem. Assume that all eigenvalues are real.

1. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(\pi) = 0.$
2. $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(L) = 0.$

In each of the following problems, find the Fourier series corresponding to the given function:

3. $f(x) = \begin{cases} 1, & -l \leq x < 0, \\ 0, & 0 \leq x < l, \end{cases} \quad f(x+2l) = f(x).$
4. $f(x) = x, \quad -1 \leq x < 1, \quad f(x+2) = f(x).$
5. $f(x) = \begin{cases} x+1, & -1 \leq x < 0, \\ x, & 0 \leq x < 1, \end{cases} \quad f(x+2) = f(x).$
6. $f(x) = \begin{cases} x+1, & -1 \leq x < 0, \\ 1-x, & 0 \leq x < 1, \end{cases} \quad f(x+2) = f(x).$
7. $f(x) = \begin{cases} x+l, & -l \leq x < 0, \\ l, & 0 \leq x < l, \end{cases} \quad f(x+2l) = f(x).$

8. If $f(x) = -x$ for $-l < x < l$ and $f(x+2l) = f(x)$, find a formula for $f(x)$ in the interval $l < x < 2l$, and in the interval $-3l < x < -2l$.

Find the Fourier series for the following problems. Assume that the functions are periodically extended outside the original interval. Sketch the function to which each series converges outside the original interval.

9. $f(x) = \begin{cases} 1, & 0 \leq x < s < 1, \\ 0, & s \leq x < 2-s, \\ 1, & 2-s \leq x < 2. \end{cases}$
10. $f(x) = 1 - x^2, \quad -1 < x < 1,$
11. $f(x) = \begin{cases} 0, & -1 \leq x < 0, \\ x^2, & 0 \leq x < 1. \end{cases}$

In each of the following problems find the required Fourier series for the given function; sketch the graph of the function to which the function converges over two or three periods.

12. $f(x) = \begin{cases} 1-x, & 0 < x \leq 1, \\ 0, & 1 < x \leq 2, \end{cases}$ Both even and odd extensions, period 4.
13. $f(x) = \begin{cases} x, & 0 \leq x < 1, \\ 1, & 1 \leq x < 2, \end{cases}$ Sine series, period 4.
14. $f(x) = 1, \quad 0 \leq x \leq \pi$ Cosine series, period 2π .
15. $f(x) = 1, \quad 0 < x < \pi$ Sine series, period 2π .
16. $f(x) = l - x, \quad 0 \leq x \leq l$ Cosine series, period $2l$.
17. $f(x) = l - x, \quad 0 < x < \pi$ Sine series, period $2l$.

18. State exactly the boundary value problem determining the temperature in a silver rod 2 meters long if the ends are held at the temperatures 30° Celsius and 50° Celsius, respectively. The thermal diffusivity of silver is $\alpha^2 = 1.71 \text{ cm}^2/\text{sec}$. Assume that the initial temperature in the bar is given by a quadratic function of the distance along the bar consistent with the preceding boundary conditions, and with the condition that the temperature at the center of the rod is 60° Celsius.

19. Find the solution of the heat conduction problem

$$\begin{aligned} u_{xx} &= 4u_t, & 0 < x < 2, & \quad t > 0, \\ u(0, t) &= 0, & u(2, t) &= 0, & \quad t > 0, \\ u(x, 0) &= 2 \sin \frac{\pi x}{2} - \sin \pi x + 4 \sin 2\pi x. \end{aligned}$$

In each of the following problems determine whether the method of separation of variables can be used to replace the given partial differential equation by a pair of ordinary differential equations. If so, find the equations.

20. $u_{xx} + u_{xt} + u_t = 0,$ 21. $xu_{xx} + u_t = 0,$ 22. $u_{xx} + (x + y)u_{yy} = 0.$

23. Consider the conduction of heat in a copper rod 100 cm in length whose ends are maintained at 0° Celsius for all $t > 0$. Find an expression for the temperature $u(x, t)$ if the initial temperature distribution in the rod is given by

$$u(x, 0) = \begin{cases} 0, & 0 \leq x < 25 \\ 50, & 25 \leq x \leq 75 \\ 0, & 75 < x \leq 100 \end{cases}$$

24. Let a metallic rod 20 cm long be heated to a uniform temperature of 100° Celsius. Suppose that at $t = 0$ the ends of the bar are plunged into an ice bath at 0° Celsius, and thereafter maintained at this temperature, but that no heat is allowed to escape through the lateral surface. Find an expression for the temperature at any point in the bar at any later time. Use two terms in the series expansion to determine approximately the temperature at the center of the bar at time $t = 30$ sec if the bar is made of (a) silver, (b) aluminum, or

(c) cast iron. Thermal diffusivities of silver, aluminum, and cast iron are $1.71 \text{ cm}^2/\text{sec}$, $0.86 \text{ cm}^2/\text{sec}$, and $0.12 \text{ cm}^2/\text{sec}$, respectively. Also, use just one term in the series expansion for $u(x, t)$ to find the time that will elapse before the center of the bar cools to a temperature of 25° Celsius for each of the three metals.

25. Let an aluminum rod of length l be initially at the uniform temperature of 25° Celsius . Suppose that at time $t = 0$ the end $x = 0$ is cooled to 0° Celsius , while the end $x = l$ is heated to 60° Celsius , and both are thereafter maintained at those temperatures.

- Find the temperature distribution in the rod at any time t . Assume in the remaining parts of this problem that $l = 20 \text{ cm}$.
- Use only the first term in the series for the temperature $u(x, t)$ to find the approximate temperature at $x = 5 \text{ cm}$ when $t = 30 \text{ sec}$; when $t = 60 \text{ sec}$.
- Use the first two terms in the series for $u(x, t)$ to find an approximate value of $u(5, 30)$. What is the percentage difference between the one- and two-term approximations? Does the third term have any appreciable effect for this value of t ?
- Use the first term in the series for $u(x, t)$ to estimate the time interval that must elapse before the temperature at $x = 5 \text{ cm}$ comes within 1 percent of its steady state value.

26. Consider a uniform rod of length l with an initial temperature given by $\sin(\pi x/l)$, $0 \leq x \leq l$. Assume that both ends of the bar are insulated. Find the temperature $u(x, t)$, and the steady-state temperature as $t \rightarrow \infty$.

In each of the following problems find the steady-state solution of the heat conduction equation $u_t = \alpha^2 u_{xx}$ that satisfies the given set of boundary conditions.

$$27. \quad u_x(0, t) = 0, \quad u(l, t) = T. \quad 28. \quad u(0, t) = 30, \quad u(40, t) = -20.$$

$$29. \quad u(0, t) = T, \quad u_x(l, t) = 0.$$

30. Find the steady-state solution of the partial differential equation

$$u_{xx} - u_x + u_t = 0$$

that satisfies the boundary conditions

$$u(0, t) = 0, \quad u_x(l, t) = e.$$

31. Consider a uniform bar of length l having an initial temperature distribution given by $f(x)$, $0 \leq x \leq l$. Assume that the temperature at the end $x = 0$ is held at 0° Celsius , while the end $x = l$ is insulated so that no heat passes through it.

- Show that the fundamental solutions of the partial differential equation and boundary conditions are

$$u_n(x, t) = e^{-(2n-1)^2 \pi^2 \alpha^2 t / 4l^2} \sin[(2n-1)\pi x / 2l], \quad n = 1, 2, 3 \dots$$

(b) Find a formal series expansion for the temperature $u(x, t)$,

$$u(x, t) = \sum_{n=1}^{\infty} c_n u_n(x, t),$$

that also satisfies the initial condition $u(x, 0) = f(x)$.

32. Find the displacement $u(x, t)$ in an elastic string, fixed at both ends, that is set in motion with no initial velocity from the initial position $u(x, 0) = f(x)$, where

$$f(x) = \begin{cases} Ax & 0 \leq x \leq l/4, \\ Al/4, & l/4 < x < 3l/4, \\ A(l-x), & 3l/4 \leq x \leq l. \end{cases}$$

33. Find the displacement $u(x, t)$ in an elastic string of length l , fixed at both ends, that is set in motion from its straight equilibrium position with the initial velocity g defined by

$$f(x) = \begin{cases} Ax & 0 \leq x \leq l/2, \\ A(l-x), & l/2 < x < l. \end{cases}$$

34. If an elastic string is free at one end, the boundary condition there is that $u_x = 0$. Find the displacement in an elastic string of length l , fixed at $x = 0$ and free at $x = l$, set in motion with no initial velocity from the initial position $u(x, 0) = f(x)$, where f is a given function. HINT: Show that the fundamental solutions for this problem, satisfying all conditions except the inhomogeneous initial condition, are

$$u_n(x, t) = \sin[(2n-1)\pi x/2l] \cos[(2n-1)\pi at/2l], \quad n = 1, 2, 3, \dots$$

35. A vibrating string moving in an elastic medium satisfies the equation

$$a^2 u_{xx} - \alpha^2 u = u_{tt},$$

where α^2 is proportional to the coefficient of elasticity in the medium. Suppose that the string is fixed at both ends, and is released with no initial velocity from the initial position $u(x, t) = f(x)$, $0 < x < l$. Find the displacement $u(x, t)$.

36. The motion of a circular elastic membrane, such as a drum head, is governed by the two-dimensional wave equation in polar coordinates

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = \frac{1}{a^2} u_{tt}.$$

Assuming that $u(r, \theta, t) = R(r)\Theta(\theta)T(t)$, find ordinary differential equations satisfied by $R(r)$, $\Theta(\theta)$, and $T(t)$. (Do not solve them.)