MATH-2400 Sections 17–20 Name: ______________________

Section: ______________________

Instructor: Joe Klobusicky

Exam #3

Please show all work.

Do not use text books, notes, calculators, or other aids.

You may use both sides of one $8\frac{1}{2} \times 11$" crib sheet.

PROBLEM #	POINTS
1 (40 pts)	
2 (18 pts)	
3 (26 pts)	
4 (16 pts)	
Total	

1. Let $u(x,t)$ describe temperature in a bar of unobtanium having thermal diffusivity of 1 cm^2/s and length $10cm$. Suppose both sides at all positive times maintain a temperature of 0 degrees Celsius. In this case, $u(x,t)$ satisfies the heat equation

$$u_t = u_{xx},$$

with boundary conditions

$$u(0,t) = 0, \qquad u(10,t) = 0.$$

The bar has an initial temperature profile of

$$u(x,0) = \begin{cases} 10 & 0 < x < 5, \\ 0 & 5 \le x < 10. \end{cases}$$

a) [16 pts.] Assuming a separation of variables $u(x,t) = X(x)T(t)$, derive ODEs for $X(x)$ and $T(t)$, and boundary values for $X(x)$. (You do not need to solve these ODEs).

b) [16 pts.] The general solution for the problem described above is

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-(n\pi/10)^2 t} \sin(n\pi x/10).$$

Compute the particular solution corresponding to the initial conditions given above.

c)[8 pts.] Sketch approximate solutions for $u(x,.000001)$ and $u(x,10^6)$.

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2. Consider the eigenvalue problem

$$X'' + \sigma X = 0, \qquad X(-1) = 0, \qquad X'(0) = 0.$$

a)[10 pt] Are there positive eigenvalues? If so, what are the corresponding eigenfunctions?

b)[8 pt] Is $\sigma = 0$ an eigenvalue? If so, what is the corresponding eigenfunction?

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3. [20 pts.] (a)Compute the **cosine expansion** for the function $f(x) = \sin(x)$ defined on $0 < x < \pi$.

Hint: $2\sin(a)\cos(b) = \sin(a+b) + \sin(a-b)$.

(b) [6 pt] Sketch what this expansion will look like in the region $-2\pi < x < 2\pi$.

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4. Consider the following PDE used in **renewal theory** describing waiting times $u(x,t)$ for customers in a line:

$$u_t(x,t) - u_x(x,t) = u(0,t)p(x). \quad x, t \geq 0. \tag{1}$$

Here $p(x)$ is some positive function defined on $[0, \infty)$.

(a)[16 pt] Assuming a solution of the form $u(x,t) = X(x)T(t)$, find ordinary differential equations satisfied by $X(x)$ and $T(t)$. **Note**: do not attempt to solve these equations!

Bonus [3 pt] If $p(x)$ satisfies $\int_0^\infty p(x)dx = 1$ and $u(x,t) \to 0$ as $x \to \infty$, show that the total number $N(t) = \int_0^\infty u(x,t)dx$ is conserved, meaning that $N(t) = N(0)$ for all $t > 0$.