
MATH 2400: Suggested Homework 2

Find the solutions of the given differential equations:

1. $y' = \frac{x^2}{y}$

2. $y' + y^2 \sin x = 0$

3. $xy' = \sqrt{1 - y^2}$

Find the **explicit** solutions to the initial value problems:

4. $y' = \frac{x(x^2 + 1)}{4y^3}, \quad y(0) = -\frac{1}{\sqrt{2}}$

5. $y' = \frac{x^2}{y(1 + x^3)}, \quad y(0) = -1$

6. Einsteinium-253 decays at a rate proportional to the amount present. Determine the half-life, τ , if this material loses one third of its mass in 11.7 days.

7. Suppose that 100 mg of thorium-234 are initially present in a closed container, and that thorium-234 is added to the container at a constant rate of 1 mg/day.

- Find the amount $Q(t)$ of thorium-234 in the container at any time, given that its decay rate is $0.02828 \text{ days}^{-1}$.
- Find the limiting amount Q_1 of thorium-234 in the container as $t \rightarrow \infty$.
- How long a time period must elapse before the amount of thorium-234 in the container drops to within 0.5 mg of the limiting value Q_1 ?
- If thorium-234 is added to the container at a rate of k mg/day, find the value of k that is required to maintain a constant level of 100 mg of thorium-234.

8. A tank initially contains 120 liters of pure water. A mixture containing γ g/liter of salt enters the tank at a rate 2 liters/min, and the well-stirred mixture leaves the tank at the same rate. Find an expression in terms of γ for the amount of salt in the tank at any time t . Also, find the limiting amount of salt in the tank as $t \rightarrow \infty$.

9. Suppose that a room containing 1200 ft^3 of air is originally free of carbon monoxide. Beginning at time $t = 0$ cigarette smoke, containing 4% of carbon monoxide, is introduced to the room at a rate of $0.1 \text{ ft}^3/\text{min}$, and the well-circulated mixture is allowed to leave the room at the same rate.

- Find an expression for the concentration $x(t)$ of carbon monoxide in the room at any time $t > 0$.

- (b) Extended exposure to a carbon monoxide concentration as low as 0.00012 is harmful to the human body. Find the time τ at which this concentration is reached.

In each of the following two problems sketch dN/dt versus N , determine the equilibrium points and classify each one as stable or unstable:

$$10. \quad \frac{dN}{dt} = aN + vN^2, \quad a, b > 0 \qquad 11. \quad \frac{dN}{dt} = N(N-1)(N-2)$$

12. Semistable Equilibrium Solutions: Sometimes a constant equilibrium solution has the property that solutions lying on one side of the equilibrium solution tend to approach it, whereas solutions lying on the other side recede from it. In this case the equilibrium solution is said to be **semistable**.

- (a) Consider the equation

$$\frac{dN}{dt} = k(1-N)^2 \qquad (1)$$

where k is a positive constant. Show that $N = 1$ is the only equilibrium point, with the corresponding equilibrium solution $\phi(t) = 1$.

- (b) Sketch dN/dt versus N . Show that N is increasing as a function of t for $N < 1$ and also for $N > 1$. Thus solutions below the equilibrium solution approach it while those above it grow further away. Thus $\phi(t) = 1$ is semistable.
- (c) Solve Equation (1) subject to the initial condition $N(0) = N_0$, and confirm the conclusion reached in part (b).

In each of the following two problems sketch dN/dt versus N and determine the equilibrium points. Also classify each equilibrium point as stable, unstable, or semistable:

$$13. \quad \frac{dN}{dt} = N(1-N^2), \qquad 14. \quad \frac{dN}{dt} = N^2(4-N^2)$$

15. Consider the following model of a fishery. Let us assume that the fish are caught at a constant rate h independent of the size of the fish population $N(t)$. Then N satisfies the equation

$$\frac{dN}{dt} = r \left(1 - \frac{N}{K} \right) N - h, \qquad (2)$$

where r and K are some positive constants.

- (a) If $h < rK/4$, show that (2) has two equilibrium points N_1 and N_2 with $N_1 < N_2$; determine these points.
- (b) Show that N_1 is unstable and N_2 is stable.
- (c) From a plot of dN/dt versus N show that if the initial population $N_0 > N_1$, then $N(t) \rightarrow N_2$ as $t \rightarrow \infty$, but if $N_0 < N_1$, then $N(t)$ decreases as t increases. Note that $N = 0$ is not an equilibrium point, so if $N_0 < N_1$, the extinction will be reached in a finite time.
- (d) If $h > rk/4$, show that $N(t)$ decreases to zero as t increases regardless of the value of N_0 .
- (e) If $h = rK/4$, show that there is a single equilibrium point $N = K/2$, and that this equilibrium point is semistable. Notice that $h_m = rK/4$ is the maximal sustainable yield of the fishery corresponding to the equilibrium value of $N = K/2$. The fishery is considered overexploited if N is reduced to a level below $K/2$.