MATH-2400 Sections 17–18 NAME: ______________________

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Exam #3

Please show all work. Each question is worth 15 points.
Do not use text books, notes, calculators, or other aids.
You may use both sides of one $8\frac{1}{2} \times 11$" crib sheet.

PROBLEM #	POINTS
1	
2	
3	
4	
Total	

1. [15 pts.] Given that the eigenvalues and eigenfunctions for the problem

$$X'' + \sigma X = 0, \qquad X'(0) = 0, \quad X'(\pi) = 0$$

are

$$\sigma_n = n^2, \qquad X_n(x) = \cos(nx), \qquad n = \overset{0}{\cancel{1}}, 2, \ldots,$$

use separation of variables to find the functions of $u_n(x,t)$ that satisfy the heat equation

$$u_t = u_{xx}, \qquad 0 < x < \pi, \quad t > 0,$$

and the boundary conditions

$$u(0,t) = 0, \quad u(\pi,t) = 0, \qquad t > 0.$$

Then, use those functions to find the full solution $u(x,t)$ to the heat equation subject to the initial condition $u(x,0) = 3 - \cos(x) + 10\cos(7x)$.

$u(x,t) = X(x)T(t)$ $\quad XT' = TX'' \Rightarrow \frac{T'}{T} = \frac{X''}{X} = -\sigma \Rightarrow T' + \sigma T = 0,\ X'' + \sigma X = 0$

B.C. $u_x(0,t) = X'(0)T(t) = 0 \Rightarrow X'(0) = 0,\ u_x(\pi,t) = X'(\pi)T(t) = 0 \Rightarrow X'(\pi) = 0.$

Eval $\sigma_n = n^2$ $\quad n = 0, 1, \ldots$ $\quad X_n(x) = \cos(nx)$

$T_n' + n^2 T_n = 0 \Rightarrow T_n(t) = c_n e^{-n^2 t}$ $\quad u_n(x,t) = e^{-n^2 t}\cos(nt)$

Gen. Sol'n $u(x,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n e^{-n^2 t}\cos(nx)$

$t = 0$ $\quad u(x,0) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(nx) = 3 - \cos(x) + 10\cos(7x)$

$c_0 = 6,\ c_1 = -1,\ c_7 = 10 \Rightarrow u(x,t) = 3 - e^{-t}\cos(x) + 10 e^{-49t}\cos(7x).$

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2. [15 pts.] Consider the initial/boundary value PDE

$$z_{xx} + \cos(x)z = z_t, \qquad 0 < x < l, \qquad t > 0,$$

$$z_x(0,t) = 0, \quad z(l,t) = 0, \qquad t > 0.$$

$$z(x,0) = x^3, \qquad 0 < x < l.$$

(a) [10 pts.] Assuming a solution of the form $z(x,t) = X(x)T(t)$, find ordinary differential equations satisfied by $X(x)$ and $T(t)$.

$TX'' + \cos(x)XT = XT'$

$\Rightarrow \frac{X''}{X} + \cos(x) \cdot = \frac{T'}{T} = \lambda \Rightarrow X'' + \cos(x) \cdot X = \lambda X$

$T' = \lambda T$

(b) [5 pts.] One of the two equations you derived in part (a) corresponds to an eigenvalue problem. Use the boundary conditions for $z(x,t)$ to derive the boundary conditions for this eigenvalue problem, and state this problem (i.e. give the boundary value problem, but you don't have to solve it).

$X'' + \cos(x)X = \lambda X$

$z_x(0,t) = X'(0)T(t) = 0$ $\qquad$ $z(l,t) = 0 \Rightarrow X(l)T(t) = 0$

$\Rightarrow X'(0) = 0.$ $\qquad$ $X(l) = 0$

3. [15 pts.] Find the displacement $u(x,t)$ of an elastic string of length l that is fixed at its ends and is set in motion by pulling the string at its center and releasing. In this case, $u(x,t)$ satisfies the wave equation

$$u_{tt} = a^2 u_{xx},$$

with the boundary conditions

$$u(0,t) = 0, \qquad u(l,t) = 0,$$

and the initial condition

$$u(x,0) = f(x), \qquad u_t(x,0) = 0,$$

with $f(x)$ defined by

$$f(x) = \begin{cases} x, & 0 \le x \le l/2, \\ l - x, & l/2 \le x \le l. \end{cases}$$

(Hint: Here, you may use the fact that

$$u(x,t) = \sum_{n=1}^{\infty} (c_n \sin(n\pi at/l) + d_n \cos(n\pi at/l)) \sin(n\pi x/l)$$

is a general solution to the wave equation under fixed endpoints.)

Since $u_t = \sum_{n=1}^{\infty}\left(\frac{n\pi a}{l} c_n \cos\left(\frac{n\pi a t}{l}\right) - \frac{n\pi a}{l} d_n \sin\left(\frac{n\pi a t}{l}\right)\right)\sin(n\pi x/l)$

$u_t(x,0) = \sum_{n=1}^{\infty} \frac{n\pi a}{l} c_n \sin\left(\frac{n\pi x}{l}\right) = 0 \Rightarrow c_n = 0.$

$u(x,0) = \sum_{n=1}^{\infty} d_n \sin(n\pi x/l) \quad d_n = \frac{2}{l}\int_0^l \sin(n\pi x/l) f(x)\,dx$

$d_n = \frac{2}{l}\left(\int_0^{l/2} x \sin\left(\frac{n\pi x}{l}\right)dx + \int_{l/2}^{l} (l-x)\sin\left(\frac{n\pi x}{l}\right)\right)$

$= \frac{2}{l}\left(\frac{l}{n\pi} x\left(-\cos\left(\frac{n\pi x}{l}\right)\right)\Big|_0^{l/2} + \frac{l}{n\pi}(l-x)\left(-\cos\left(\frac{n\pi x}{l}\right)\right)\Big|_{l/2}^{l} + \frac{l}{n\pi}\int_0^{l/2} \cos\left(\frac{n\pi x}{l}\right)dx - \frac{l}{n\pi}\int_{l/2}^{l} \cos\left(\frac{n\pi x}{l}\right)dx\right)$

$= \frac{2}{n\pi}\left(\frac{-l}{2}\cos(n\pi/2) + \frac{l}{2}\cos\left(\frac{n\pi}{2}\right) + \left(\frac{l}{n\pi}\right)\sin\left(\frac{n\pi x}{l}\right)\Big|_0^{l/2} - \left(\frac{l}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{l}\right)\Big|_{l/2}^{l}\right)$

$= \frac{2}{n\pi}\left(\frac{l}{n\pi}\left(\sin\left(\frac{n\pi}{2}\right) + \sin\left(\frac{n\pi}{2}\right)\right)\right) \Rightarrow u(x,t) = \frac{4l}{\pi^2}\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{2}\right)}{n^2} \cdot \cos\left(\frac{n\pi a t}{l}\right)\sin\left(\frac{n\pi x}{l}\right)$

$= \frac{4l}{\pi^2 n^2} \sin\left(\frac{n\pi}{2}\right)$

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4. [15 pts.] Find all the eigenvalues and eigenfunctions of the boundary value problem

$$X'' + \sigma X = 0, \qquad X'(0) = 0, \qquad X(5) = 0$$

$\sigma < 0 \Rightarrow \sigma = -\mu^2 \Rightarrow r_{1,2} = \pm\mu \quad X(t) = C_1 e^{-\mu x} + C_2 e^{\mu x}$

$X'(0) = -\mu C_1 + \mu C_2 = 0 \Rightarrow C_2 - C_1 = 0$

$X(5) = C_1 e^{-5\mu} + C_2 e^{5\mu} = 0$

$-C_1 + C_2 = 0$

$e^{-5\mu} C_1 + e^{5\mu} C_2 = 0$

$\begin{vmatrix} -1 & 1 \\ e^{-5\mu} & e^{5\mu} \end{vmatrix} = -e^{5\mu} - e^{-5\mu} < 0.$

Only Sol. is $C_1 = C_2 = 0 \Rightarrow$ No E-val.

$\sigma = 0 \quad X'' = 0 \Rightarrow X = ax + b$, $X'(0) = a = 0$, so $X = b$

$b = X(5) = 0 \Rightarrow b = 0$, so no 0 eval

$\sigma > 0 \quad X(x) = C_1 \sin(\mu x) + C_2 \cos(\mu x)$

$X'(0) = \mu C_1 \cos(0) - \mu C_2 \sin(0) = \mu C_1 = 0 \Rightarrow C_1 = 0$

$X(x) = C_2 \cos(\mu x)$

$0 = X(5) = C_2 \cos(5\mu)$. Eval if $5\mu = \frac{\pi}{2} + n\pi \Rightarrow \mu = \frac{(n+\frac{1}{2})\pi}{5}$

so $\sigma_n = \left(\frac{(n+\frac{1}{2})\pi}{5}\right)^2 \quad X_n = \cos\left(\frac{(n+\frac{1}{2})\pi}{5} x\right).$

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