MATH-2400 Sections 17–20

NAME: ______________________________

Section: ______________________________

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Exam #2

Please show all work.

Do not use text books, notes, calculators, or other aids.

You may use one $8\frac{1}{2} \times 11$" crib sheet. Formulas on both sides of the sheet are permitted.

PROBLEM #	POINTS
1 (40 pts)	
2 (40 pts)	
3 (20 pts)	
Total	

1.

(a) (20 pts) Compute the general solution of the system

$$\dot{\mathbf{x}} = \begin{pmatrix} 1 & 4 \\ -2 & -5 \end{pmatrix}^{=A} \mathbf{x},$$

and sketch the phase portrait in the $x_1 - x_2$ plane. Classify the stability type of the equilibrium. What happens to the solutions as $t \to \infty$?

(b) (5 pts) Find the particular solution with initial condition $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

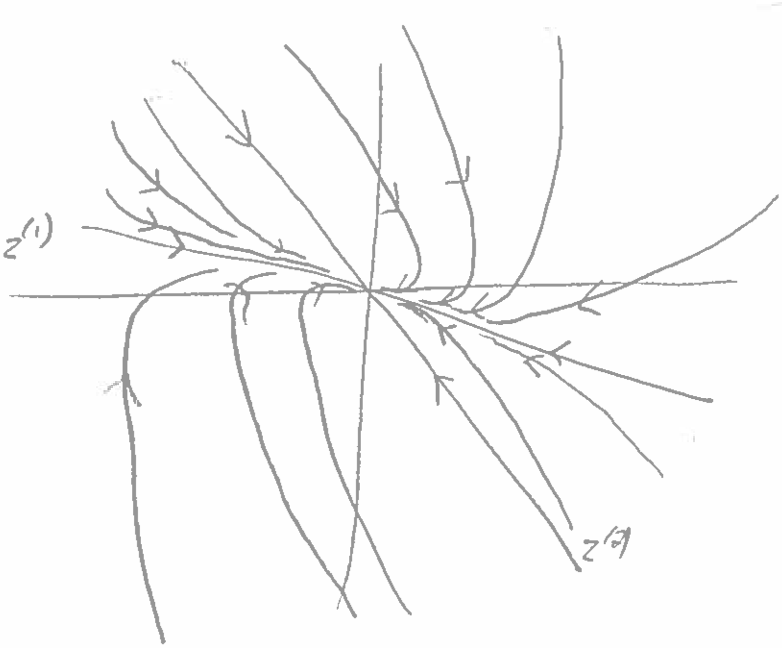
$\det(A - Ir) = \begin{vmatrix} 1-r & 4 \\ -2 & -5-r \end{vmatrix} = (1-r)(-5-r)+8 = r^2+4r+3 = (r+3)(r+1) = 0 \Rightarrow r_{1,2} = -1, -3$

$r_1 = -1$ $\begin{bmatrix} 2 & 4 \\ -2 & -4 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \vec{0} \Rightarrow 2z_1 + 4z_2 = 0 \Rightarrow z_1 = -2z_2,\ z^{(1)} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$r_2 = -3$ $\begin{bmatrix} 4 & 4 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \vec{0} \Rightarrow z_1 = -z_2,\ z^{(2)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Gen sol'n

$x(t) = c_1 e^{-t}\begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2 e^{-3t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$



Sink: Asympt. Stable. Sol'n go to origin.

For init cond.

$x(0) = c_1\begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\Rightarrow -2c_1 - c_2 = 1$

$c_1 + c_2 = 1$

$-c_1 = 2 \quad c_1 = -2,\ c_2 = 3$

So Part Sol'n

$x_p(t) = -2e^{-t}\begin{pmatrix} -2 \\ 1 \end{pmatrix} + 3e^{-3t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

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(c) (10 pts) Classify the stability type of the equilibrium for the system

$$\dot{\mathbf{x}} = \underbrace{\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}}_{=A} \mathbf{x},$$

What happens to the solutions as $t \to \infty$?

(d) (5 pts) Find the particular solution $\mathbf{x}_p(t)$ under the initial condition $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Note: This question does not require drawing the phase plane.

c) $\det(A - Ir) = (2-r)^2 + 1 = r^2 - 4r + 5$

$r_{1,2} = \frac{4 \pm \sqrt{16-20}}{2} = \underset{\mu}{2} \pm \underset{\lambda}{i}$ Since $\mu > 0$, this is an unstable spiral source. Sol'ns go to ∞ (have arbitrarily large magnitude).

d) Since $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is an equilibrium, $x_p(t) \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for all t.

2. (a) (20 pts.) For $b > 2$, an oscillating mass-spring system is modeled by the equation

$$\ddot{u}(t) + b\dot{u}(t) + u(t) = \sin t.$$

Compute the general solution of this problem. Your answer should be in terms of b, as well as two arbitrary parameters C_1 and C_2.

Comp. sol'n $r^2+br+1=0.$ $r_{1,2} = \frac{-b \pm \sqrt{b^2-4}}{2}$ (Real since $b>2$)

$u_c(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$

Part Sol'n $u_p = A\sin(t) + B\cos(t)$

$+bu_p' = Ab\cos(t) - Bb\sin(t)$

$u_p'' = -A\sin(t) - B\cos(t)$

$\sin(t) = Ab\cos(t) - Bb\sin(t)$

$\Rightarrow Ab = 0, \; -Bb = 1$

$A = 0 \quad B = -\frac{1}{b}$

$u_p(t) = -\frac{1}{b}\cos(t).$

Gen. Sol $u(t) = u_c(t) + u_p(t)$

2. (b) Consider a **high viscosity limit** of the equation in part (a), in which we make the parameter b very large.

i)(7 pt) State the steady state solution $u_s(t)$ from part (a). For any fixed $t > 0$, find $\lim_{b\to\infty} u_s(t)$. (Recall that the steady state solution is the solution that the system follows for large times, regardless of the initial conditions.)

ii)(8 pt) State the transient solution $u_t(t)$ from part (a). For any fixed $t > 0$, find $\lim_{b\to\infty} u_t(t)$. (Your answer should be dependent on the arbitrary parameters C_1, C_2 that you found in part (a))

iii)(5 pt) Are velocities of the solution $\dot{u}(x)$ very large or very small as $b \to \infty$? Explain why in words (Hint: You do not need the solution to answer this question. What does the parameter b describe?).

(i) $u_s(t) = u_p(t) = -\frac{1}{b}\cos(t) \to 0$ as $b \to \infty$.

(ii) $u_t(t) = u_c = C_1 e^{r_1 t} + C_2 e^{r_2 t}$ $\quad r_1 = \frac{-b+\sqrt{b^2-4}}{2} \cdot \frac{(-b-\sqrt{b^2-4})}{(-b-\sqrt{b^2-4})} = \frac{b^2-(b^2-4)}{-b-\sqrt{b^2-4}} = \frac{4}{-b-\sqrt{b^2-4}}$

$\to 0$ as $b \to \infty$.

$r_2 = \frac{-b-\sqrt{b^2-4}}{2} \to -\infty$ as $b \to \infty \Rightarrow C_1 e^{r_1 t} + C_2 e^{r_2 t} \to C_1 e^0 + 0 = C_1$.

(iii) As viscosity increases, the spring moves slower, thus velocities are small.

3. For unknown vectors $\mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and numbers r, consider the system

$$A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = r \begin{pmatrix} z_2 \\ z_1 \end{pmatrix}. \tag{1}$$

Warning: This is **not** the usual eigenvalue equation! Notice that the components of the vectors in equation (1) have been flipped on the right hand side!

(a) (10 pt) Rewrite equation (1) in the form

$$B \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{2}$$

where B is a 2×2 matrix dependent on r.

(b) (10 pt) Now, solve equation (2) for two distinct nonzero values of $\mathbf{z}$ and r. If you couldn't find B in part (a), use $B = \begin{pmatrix} 2-r & 3+r \\ 1 & 3-r \end{pmatrix}$ under a 2 point deduction.

a) Method 1 $A\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = r\begin{pmatrix} z_2 \\ z_1 \end{pmatrix} = r\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \Rightarrow \left(A - \begin{pmatrix} 0 & r \\ r & 0 \end{pmatrix}\right)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 0 \Rightarrow B = \begin{pmatrix} 2 & 2-r \\ 1-r & 3 \end{pmatrix}$

Method 2 Write out eqns:

$$\begin{aligned} 2z_1 + 2z_2 &= rz_2 \\ z_1 + 3z_2 &= rz_1 \end{aligned} \Rightarrow \begin{aligned} 2z_1 + (2-r)z_2 &= 0 \\ (1-r)z_1 + 3z_2 &= 0. \end{aligned} \Rightarrow \begin{pmatrix} 2 & 2-r \\ 1-r & 3 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \vec{0}$$

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b) $\det(B) = \begin{vmatrix} 2 & 2-r \\ 1-r & 3 \end{vmatrix} = 6-(2-r)(1-r) = -r^2+3r+4 = 0 \Rightarrow r^2-3r-4=0$

$(r-4)(r+1) = 0 \quad r = 4, -1.$

$r=4$

$\begin{bmatrix} 2 & -2 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = 0 \Rightarrow z_1 = z_2 \quad z^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$r=-1$ $\begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = 0 \Rightarrow 2z_1 + 3z_2 = 0 \Rightarrow z_1 = -\frac{3}{2}z_2 \quad z^{(2)} = \begin{pmatrix} -\frac{3}{2} \\ 1 \end{pmatrix}$

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