HW #5 Solutions

#1)

$$\dot{\underline{x}} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \underline{x}$$

$$\begin{vmatrix} 3-r & -2 \\ 2 & -2-r \end{vmatrix} = -(3-r)(2+r)+4 =$$

$$= -6 - r + r^2 + 4 =$$

$$= r^2 - r - 2 = (r-2)(r+1)$$

$$r_1 = -1, \quad r_2 = 2$$

$r_1 = -1$: $\underline{z}^{(1)} = \begin{pmatrix} z_1^{(1)} \\ z_2^{(1)} \end{pmatrix}$

$$4 z_1^{(1)} - 2 z_2^{(1)} = 0$$

$$z_2^{(1)} = 2 z_1^{(1)}$$

CHOOSE $z_1^{(1)} = 1 \Rightarrow z_2^{(1)} = 2$

$$\underline{z}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

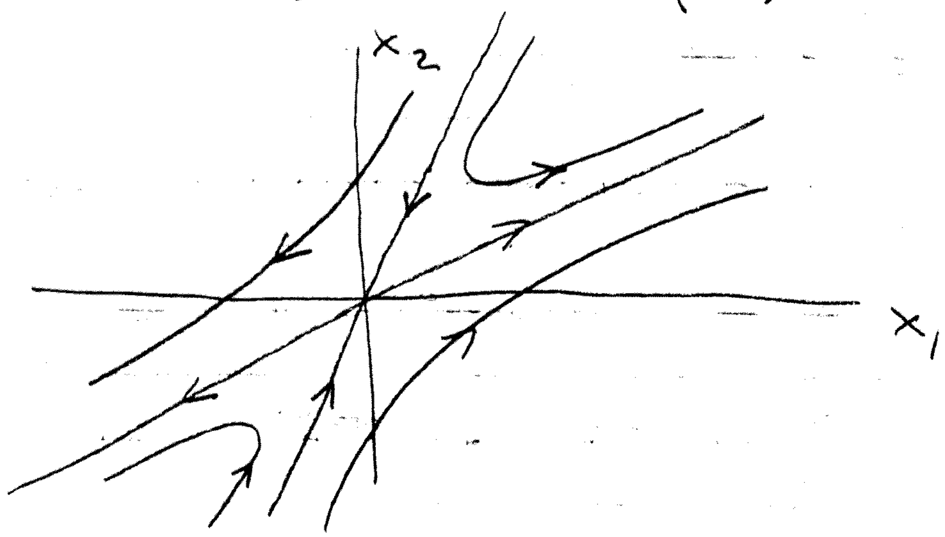
$r_2 = 2$ $\quad \underline{z}^{(2)} = \begin{pmatrix} z_1^{(2)} \\ z_2^{(2)} \end{pmatrix}$

$$z_1^{(2)} - 2 z_2^{(2)} = 0$$

CHOOSE $z_2^{(2)} = 1 \Rightarrow z_1^{(2)} = 2$

$$\underline{z}^{(2)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\underline{x}(t) = c_1\begin{pmatrix}1\\2\end{pmatrix} e^{-t} + c_2\begin{pmatrix}2\\1\end{pmatrix} e^{2t}$$



2)
$$\dot{\underline{x}} = \begin{pmatrix}1 & -2\\3 & -4\end{pmatrix}\underline{x}$$

$$\begin{vmatrix}1-r & -2\\3 & -4-r\end{vmatrix} = (1-r)(-4-r) + 6 =$$
$$= -4 + 3r + r^2 + 6 =$$
$$= r^2 + 3r + 2 = (r+2)(r+1)$$

$r_1 = -1, \quad r_2 = -2$

$\underline{r_1 = -1}$ $\quad 2z_1^{(1)} - 2z_2^{(1)} = 0$

CHOOSE $z_1^{(1)} = 1 \Rightarrow z_2^{(1)} = 1$

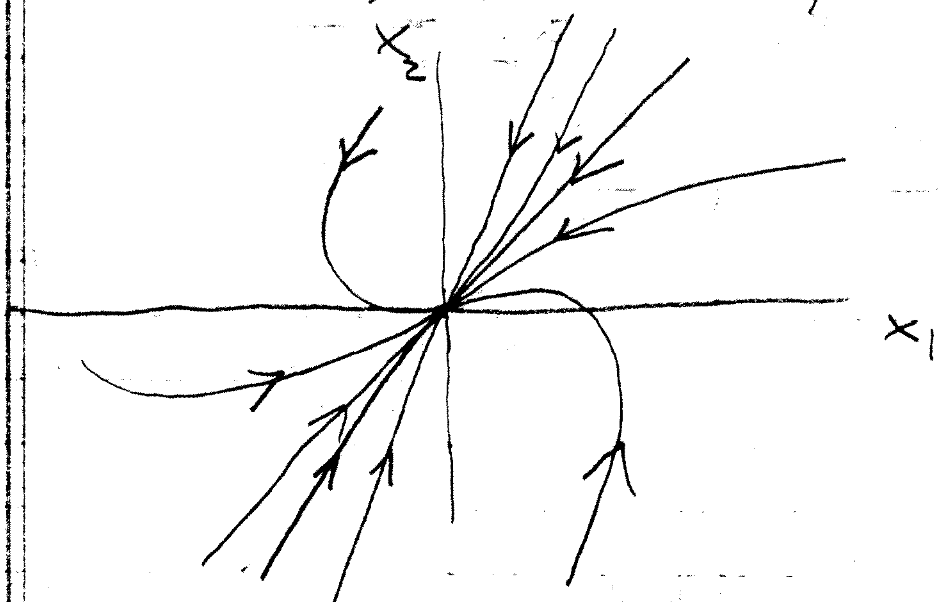
$$\underline{z}^{(1)} = \begin{pmatrix}1\\1\end{pmatrix}$$

$r_2 = -2$ $\quad 3z_1^{(2)} - 2z_2^{(2)} = 0$

CHOOSE $z_1^{(2)} = 2 \Rightarrow z_2^{(2)} = 3$

$$\underline{z}^{(2)} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$x(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{-2t}$$



3) $\underline{\dot{x}} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} x$

$$\begin{vmatrix} 1-r & 1 \\ 4 & -2-r \end{vmatrix} = (1-r)(-2-r) - 4 =$$

$$= -2 + r + r^2 - 4 =$$

$$= r^2 + r - 6 = (r-2)(r+3)$$

$r_1 = 2, \; r_2 = -3$

$\underline{r_1 = 2}$ $\quad -z_1^{(1)} + z_2^{(1)} = 0$

CHOOSE $z_1^{(1)} = 1 \Rightarrow z_2^{(1)} = 1$

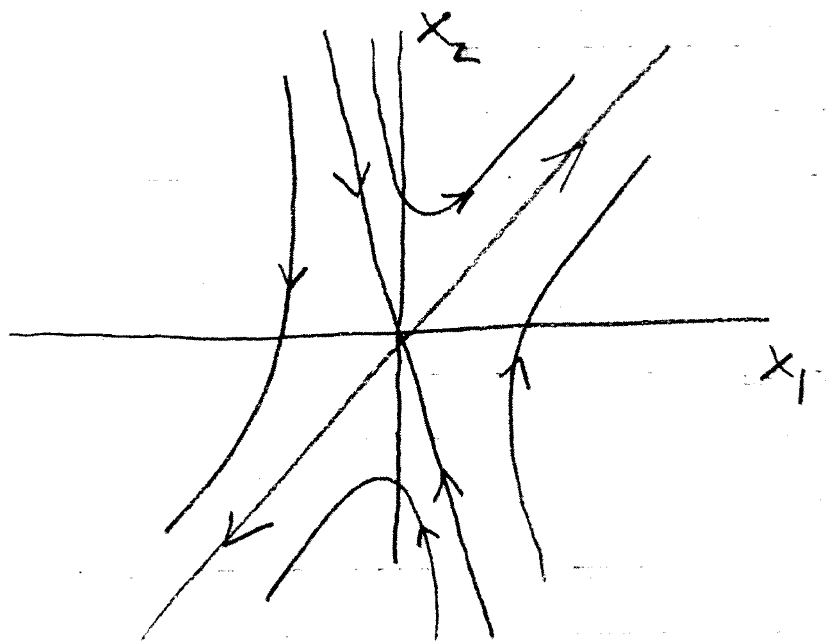
$$\underline{z}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$r_2 = -3$ $\quad 4z_1^{(2)} + z_2^{(2)} = 0$

CHOOSE $z_1^{(2)} = 1 \Rightarrow z_2^{(2)} = -4$

$$\underline{z}^{(2)} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

$$\underline{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t}$$



#4)

$$x' = \begin{pmatrix} \frac{5}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{5}{4} \end{pmatrix} x$$

$$\begin{vmatrix} \frac{5}{4}-r & \frac{3}{4} \\ \frac{3}{4} & \frac{5}{4}-r \end{vmatrix} = \left(\frac{5}{4}-r\right)^2 - \frac{9}{16} =$$

$$= \frac{25}{16} - \frac{5}{2}r - \frac{9}{16} + r^2 =$$

$$= r^2 - \frac{5}{2}r + 1 =$$

$$= \left(r - \frac{1}{2}\right)(r-2)$$

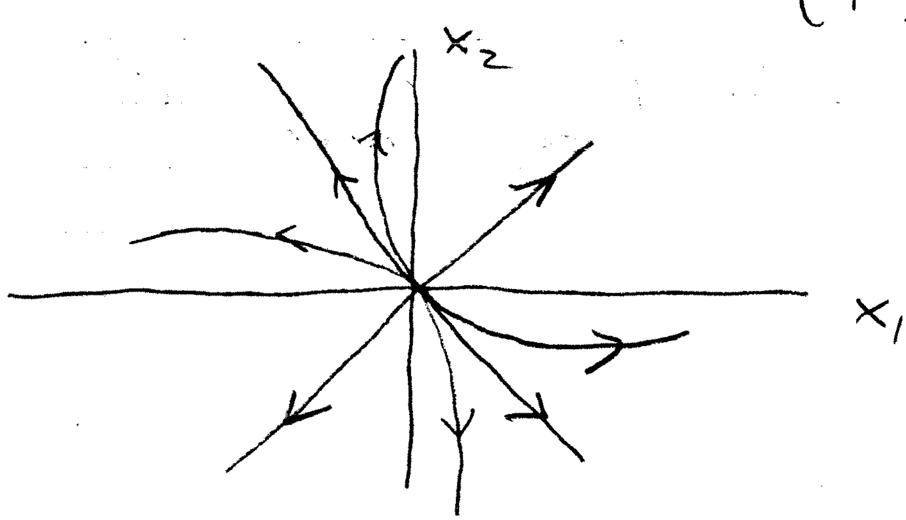
$\underline{r_1 = \frac{1}{2}}$: $\quad \frac{3}{4} z_1^{(1)} + \frac{3}{4} z_2^{(1)} = 0$

CHOOSE $z_1^{(1)} = 1 \Rightarrow z_2^{(1)} = -1 \quad z^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\underline{r_2 = 2}$: $\quad -\frac{3}{4} z_1^{(2)} + \frac{3}{4} z_2^{(2)} = 0$

CHOOSE $z_2^{(1)} = 1 \Rightarrow z_2^{(2)} = 1 \quad z^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$x(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{\frac{t}{2}} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$$



#5)

$$x' = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} x$$

$$\begin{vmatrix} 3-r & -2 \\ 4 & -1-r \end{vmatrix} = (3-r)(-1-r) + 4 =$$

$$= -3 - 2r + r^2 + 8 = r^2 - 2r + 5$$

$$r_1, r_2 = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$$

$r_1 = 1+2i$

$$(2-2i) z_1^{(1)} - 2 z_2^{(1)} = 0$$

CHOOSE: $z_1^{(1)} = 1 \Rightarrow z_2^{(1)} = 1-i$

$$\underline{z}^{(1)} = \begin{pmatrix} 1 \\ 1-i \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

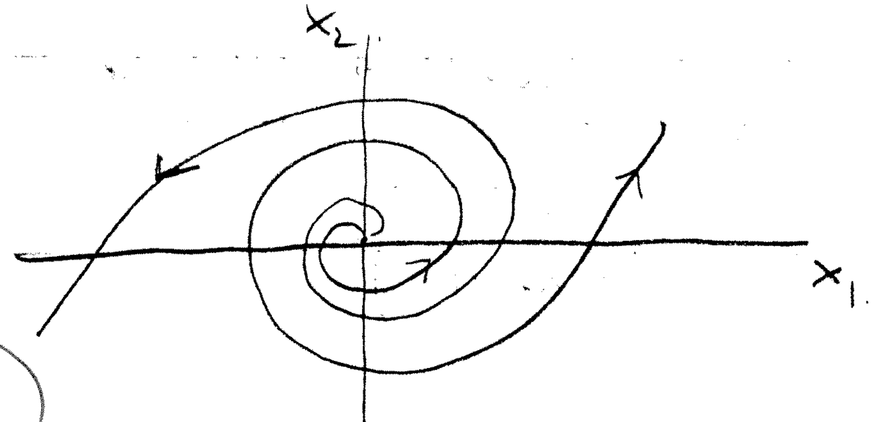
$$\underline{u}(t) = e^t \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos 2t - \begin{pmatrix} 0 \\ -1 \end{pmatrix} \sin 2t \right] =$$

$$= e^t \begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix}$$

$$\underline{v}(t) = e^t \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin 2t + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \cos 2t \right] =$$

$$= e^t \begin{pmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{pmatrix}$$

$$\underline{x}(t) = c_1 e^t \begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix} + c_2 e^t \begin{pmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{pmatrix}$$



#6)

$$\dot{\underline{x}} = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} \underline{x}$$

$$\begin{vmatrix} -1-r & -4 \\ 1 & -1-r \end{vmatrix} = (1+r)^2 + 4 = r^2 + 2r + 5$$

$$r_1, r_2 = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$$

$r_1 = -1 + 2i$

$$-2i\, z_1^{(1)} - 4 z_2^{(1)} = 0$$

CHOOSE $z_1^{(1)} = 2 \Rightarrow z_2^{(1)} = -i$

$$\underline{z}^{(1)} = \begin{pmatrix} 2 \\ -i \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

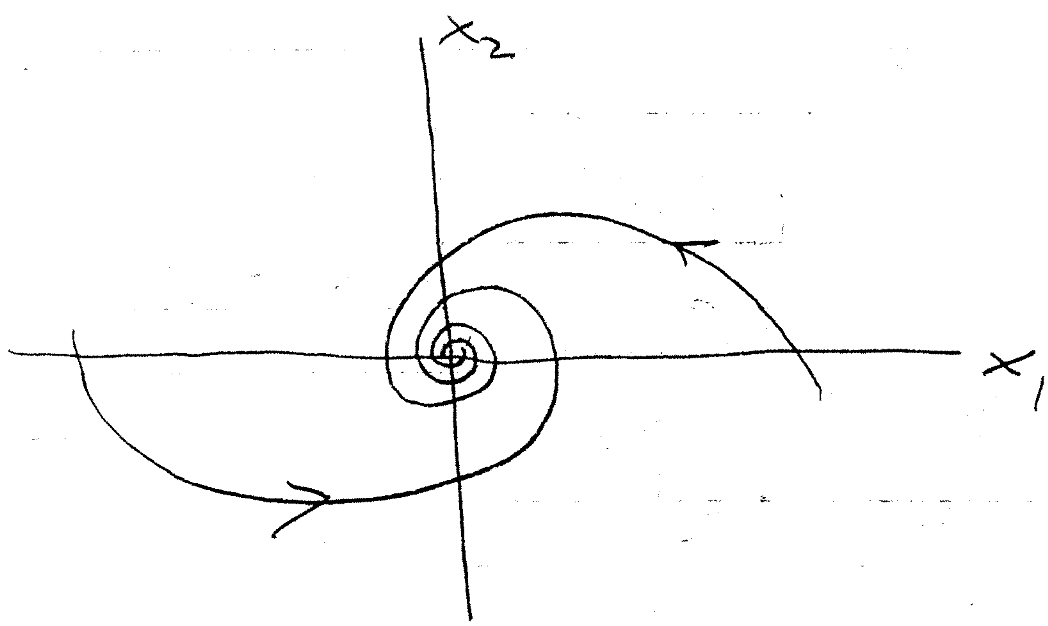
$$u(t) = e^{-t}\left[\begin{pmatrix}2\\0\end{pmatrix}\cos 2t - \begin{pmatrix}0\\-1\end{pmatrix}\sin 2t\right] =$$

$$= e^{-t}\begin{pmatrix}2\cos 2t\\ \sin 2t\end{pmatrix}$$

$$v(t) = e^{-t}\left[\begin{pmatrix}2\\0\end{pmatrix}\sin 2t + \begin{pmatrix}0\\-1\end{pmatrix}\cos 2t\right] =$$

$$= e^{-t}\begin{pmatrix}2\sin 2t\\ -\cos 2t\end{pmatrix}$$

$$x(t) = c_1 e^{-t}\begin{pmatrix}2\cos 2t\\ \sin 2t\end{pmatrix} + c_2 e^{-t}\begin{pmatrix}2\sin 2t\\ -\cos 2t\end{pmatrix}$$



#7)

$$\dot{\underline{x}} = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \underline{x}$$

(5)

$$\begin{vmatrix} 2-r & -5 \\ 1 & -2-r \end{vmatrix} = (2-r)(-2-r)+5 =$$

$$= -4+r^2+5 = r^2+1 = 0$$

$r_1 = i$, $r_2 = -i$

$\underline{r_1 = i}$: $(2-i)\, z_1^{(1)} - 5 z_2^{(1)} = 0$

$\underline{\text{CHOOSE}}$: $z_1^{(1)} = 5$, $z_2^{(1)} = (2-i)$

$$\underline{z}^{(1)} = \begin{pmatrix} 5 \\ 2-i \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

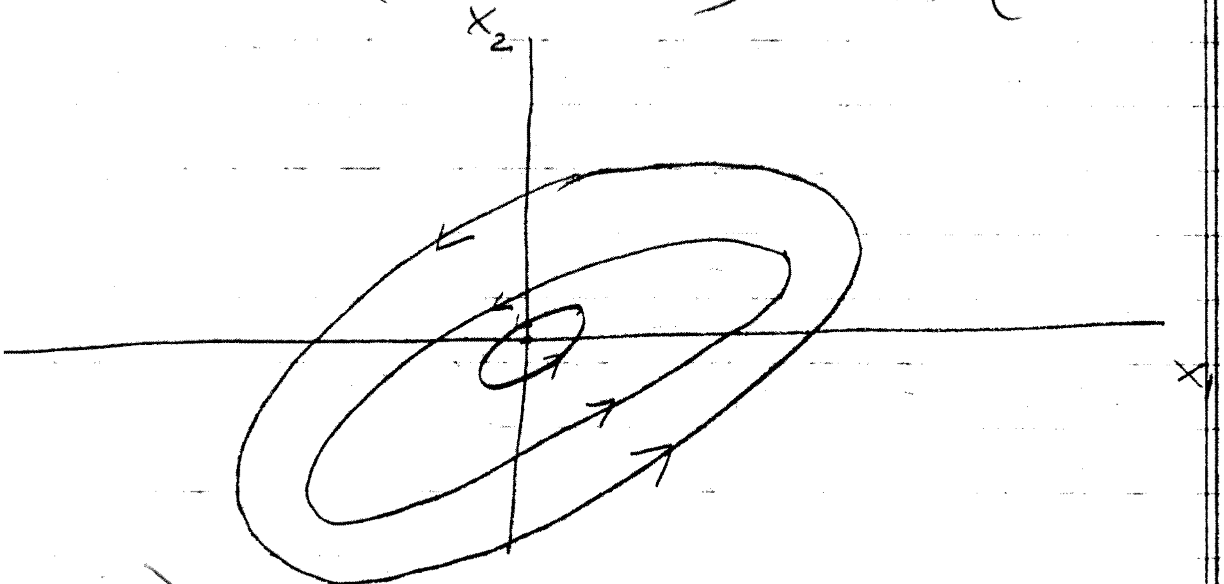
$$\underline{u}(t) = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \cos t - \begin{pmatrix} 0 \\ -1 \end{pmatrix} \sin t =$$

$$= \begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix}$$

$$\underline{v}(t) = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \sin t + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \cos t =$$

$$= \begin{pmatrix} 5\sin t \\ 2\sin t - \cos t \end{pmatrix}$$

$$\underline{x}(t) = c_1 \begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} + c_2 \begin{pmatrix} 5\sin t \\ 2\sin t - \cos t \end{pmatrix}$$



#8)

$$\dot{\underline{x}} = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix} \underline{x}$$

$$\begin{vmatrix} 1-r & 2 \\ -5 & -1-r \end{vmatrix} = (1-r)(-1-r) + 10 =$$

$$= -1 + r^2 + 10 = r^2 + 9 = 0$$

$$r_1 = 3i, \quad r_2 = -3i$$

$\underline{r_1 = 3i}$: $\quad (1-3i) z_1^{(1)} + 2 z_2^{(1)} = 0$

choose $\quad z_1^{(1)} = -2 \Rightarrow z_2 = (1-3i)$

$$\underline{z}^{(1)} = \begin{pmatrix} -2 \\ 1-3i \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} + i \begin{pmatrix} 0 \\ -3 \end{pmatrix}$$

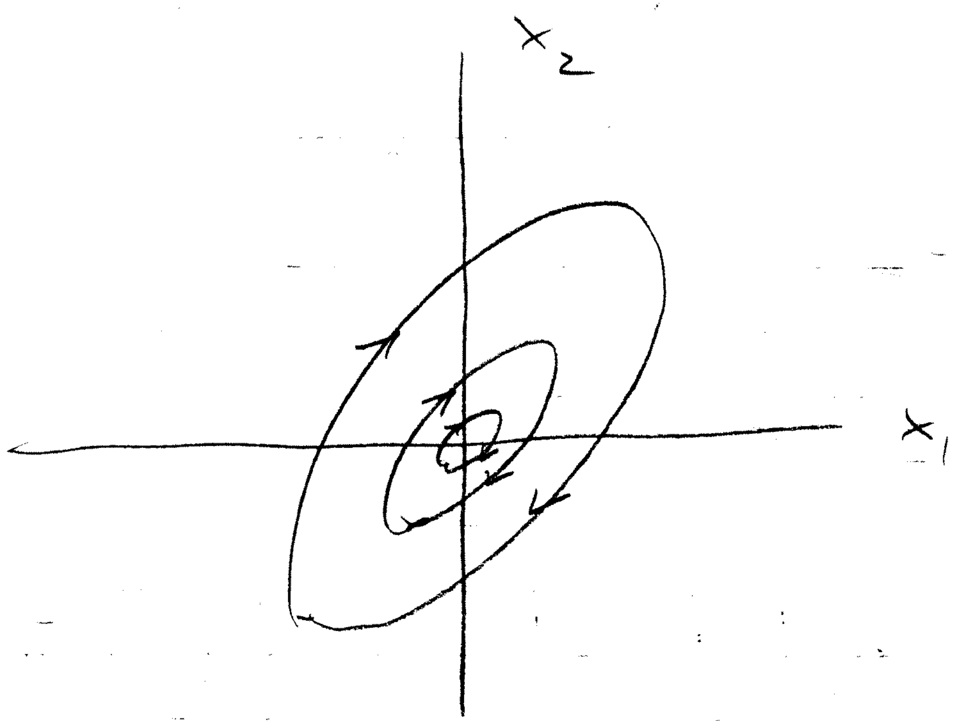
$$\underline{u}(t) = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \cos 3t - \begin{pmatrix} 0 \\ -3 \end{pmatrix} \sin 3t$$

$$= \begin{pmatrix} -2\cos 3t \\ \cos 3t + 3\sin 3t \end{pmatrix}$$

$$\underline{v}(t) = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \sin 3t + \begin{pmatrix} 0 \\ -3 \end{pmatrix} \cos 3t$$

$$= \begin{pmatrix} -2\sin 3t \\ \sin 3t - 3\cos 3t \end{pmatrix}$$

$$\underline{x}(t) = c_1 \begin{pmatrix} -2\cos 3t \\ \cos 3t + 3\sin 3t \end{pmatrix} + c_2 \begin{pmatrix} -2\sin 3t \\ \sin 3t - 3\cos 3t \end{pmatrix}$$



#9)

$$\dot{\underline{x}} = \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} x$$

$$\underline{x}' = \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \underline{x}$$

$$\begin{vmatrix} 1-r & \sqrt{3} \\ \sqrt{3} & -1-r \end{vmatrix} = (1-r)(-1-r) - 3 =$$

$$= -1 + r^2 - 3 = r^2 - 4 =$$

$$= (r-2)(r+2)$$

$\underline{r_1 = 2}$ $\quad -z_1^{(1)} + \sqrt{3}\, z_2^{(1)} = 0$

$$z_2^{(1)} = 1 \Rightarrow z_1^{(1)} = \sqrt{3}$$

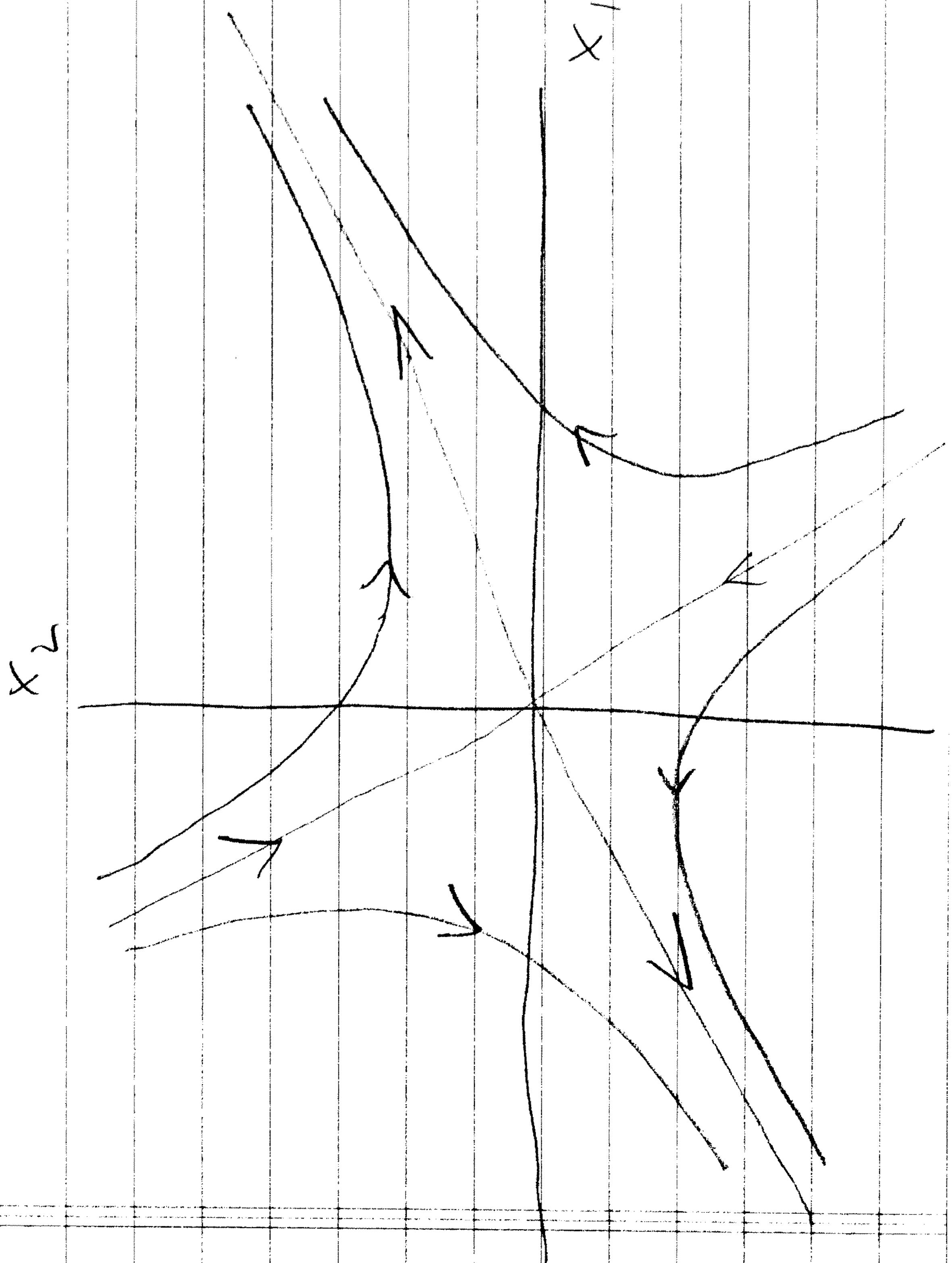
$$\underline{z}^{(1)} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$$

$\underline{r_2 = -2}$ $\quad 3 z_1^{(2)} + \sqrt{3}\, z_2^{(2)} = 0$

$$z_1^{(2)} = 1, \quad z_2^{(2)} = -\sqrt{3}$$

$$\underline{z}^{(2)} = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}$$

$$\underline{x}(t) = c_1 \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} e^{-2t}$$


x^1
x^2

10) $$\dot{x} = \overset{A}{\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}}$$

$$\det\begin{pmatrix} 3-r & -4 \\ 1 & -1-r \end{pmatrix}$$

$$= (3-r)(-1-r) + 4$$

$$= -3 - 2r + r^2 + 4$$

$$= r^2 - 2r + 1 = (r-1)^2 = 0$$

$$\Rightarrow r = 1$$

$$(A - rI)z = \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 0$$

$$\Rightarrow z_1 = +2z_2 \Rightarrow z = \begin{pmatrix} +2 \\ 1 \end{pmatrix}$$

set $z_2 = 1$

Need another eigenvector (generalized)

$$(A - rI)\eta = z$$

$$\Rightarrow \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ -\eta_2 \end{pmatrix} = \begin{pmatrix} +2 \\ 1 \end{pmatrix}$$

$$\Rightarrow \eta_1 - 2\eta_2 = 1$$

$$\Rightarrow \eta_1 = 1 + 2\eta_2$$

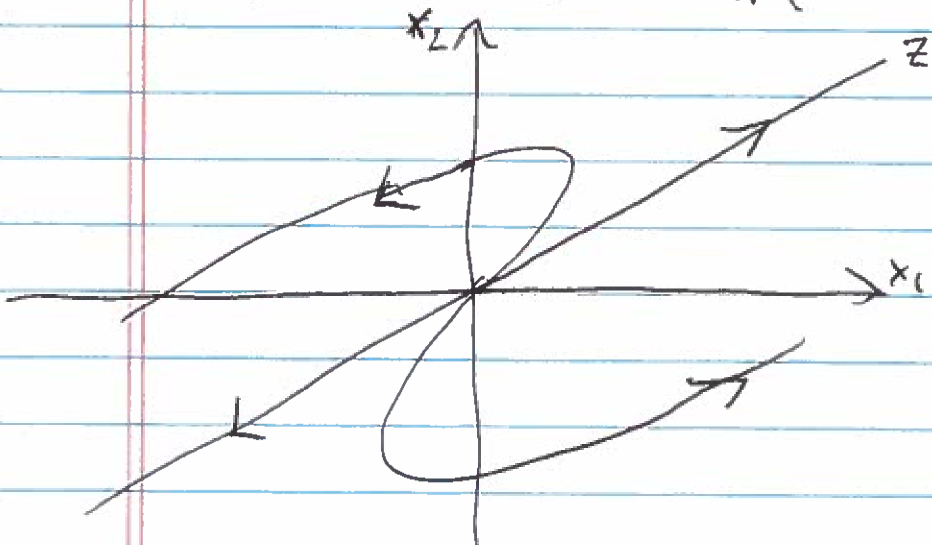
$$\Rightarrow \eta = \begin{pmatrix} 1 + 2\eta_2 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \eta_2 \underbrace{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}_{= Z}$$

part of first eigenvector

$$\Rightarrow \eta_2 = 0$$

solution

$$X = c_1 Z e^t + c_2 (Z t e^t + \eta e^t)$$



$$A \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \end{pmatrix}$$

↑ test direction of flow

11) $\dot{x} = Ax = \begin{pmatrix} -3 & 5/2 \\ -5/2 & 2 \end{pmatrix} x$

$$\det \begin{pmatrix} -3-r & 5/2 \\ -5/2 & 2-r \end{pmatrix}$$

$$= r^2 + r - 6 + \frac{25}{4}$$

$$= r^2 + r + \frac{1}{4} = \left(r + \frac{1}{2}\right)^2 = 0$$

$$\Rightarrow \quad r = -1/2$$

$$(A - rI)z = \begin{pmatrix} -5/2 & 5/2 \\ -5/2 & 5/2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 0$$

$\Rightarrow z_1 = z_2 \quad \Rightarrow \quad z = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

let $z_1 = 1$

$$(A - rI)\eta = z$$

$$\Rightarrow \begin{pmatrix} -5/2 & 5/2 \\ -5/2 & 5/2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

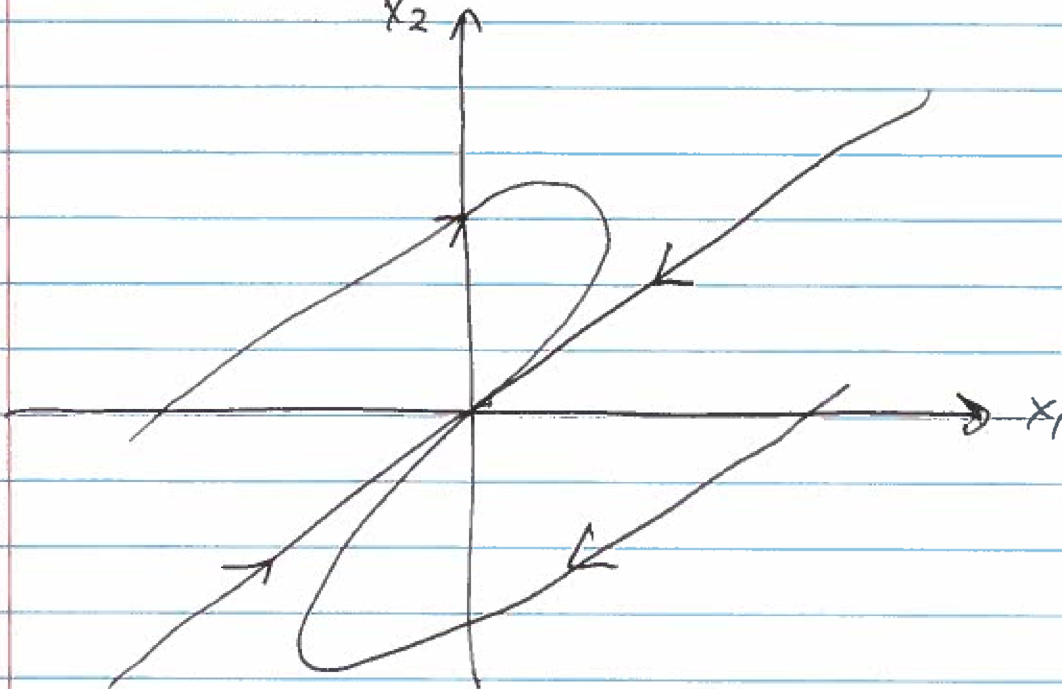
part of let

$z \Rightarrow \eta_1 = 0$

$$\Rightarrow 5\eta_2 = 2 + 5\eta_1 \Rightarrow \eta = \begin{pmatrix} 0 \\ 2/5 \end{pmatrix} + \eta_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \eta_2 = \frac{2}{5} + \eta_1$$

$$x = c_1 \xi e^{-t/2} + c_2 (\xi t e^{-t/2} + \eta e^{-t/2})$$



Directions of trajectories at points:

$$A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 5/2 \\ 2 \end{pmatrix} = \nearrow$$

$$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ -5/2 \end{pmatrix} = \swarrow$$

12) $\dot{x} = Ax = \begin{pmatrix} -3/2 & 1 \\ -1/4 & -1/2 \end{pmatrix} x$

$$\det(A - rI) = \det\begin{pmatrix} -3/2 - r & 1 \\ -1/4 & -1/2 - r \end{pmatrix}$$

$$= r^2 + 2r + 3/4 + 1/4 = r^2 + 2r + 1$$

$$= (r+1)^2 = 0 \Rightarrow r = -1$$

$$(A - rI)z = \begin{pmatrix} -1/2 & 1 \\ -1/4 & 1/2 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 0$$

$$\Rightarrow z_1 = 2z_2 \Rightarrow z = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

let $z_2 = 1$

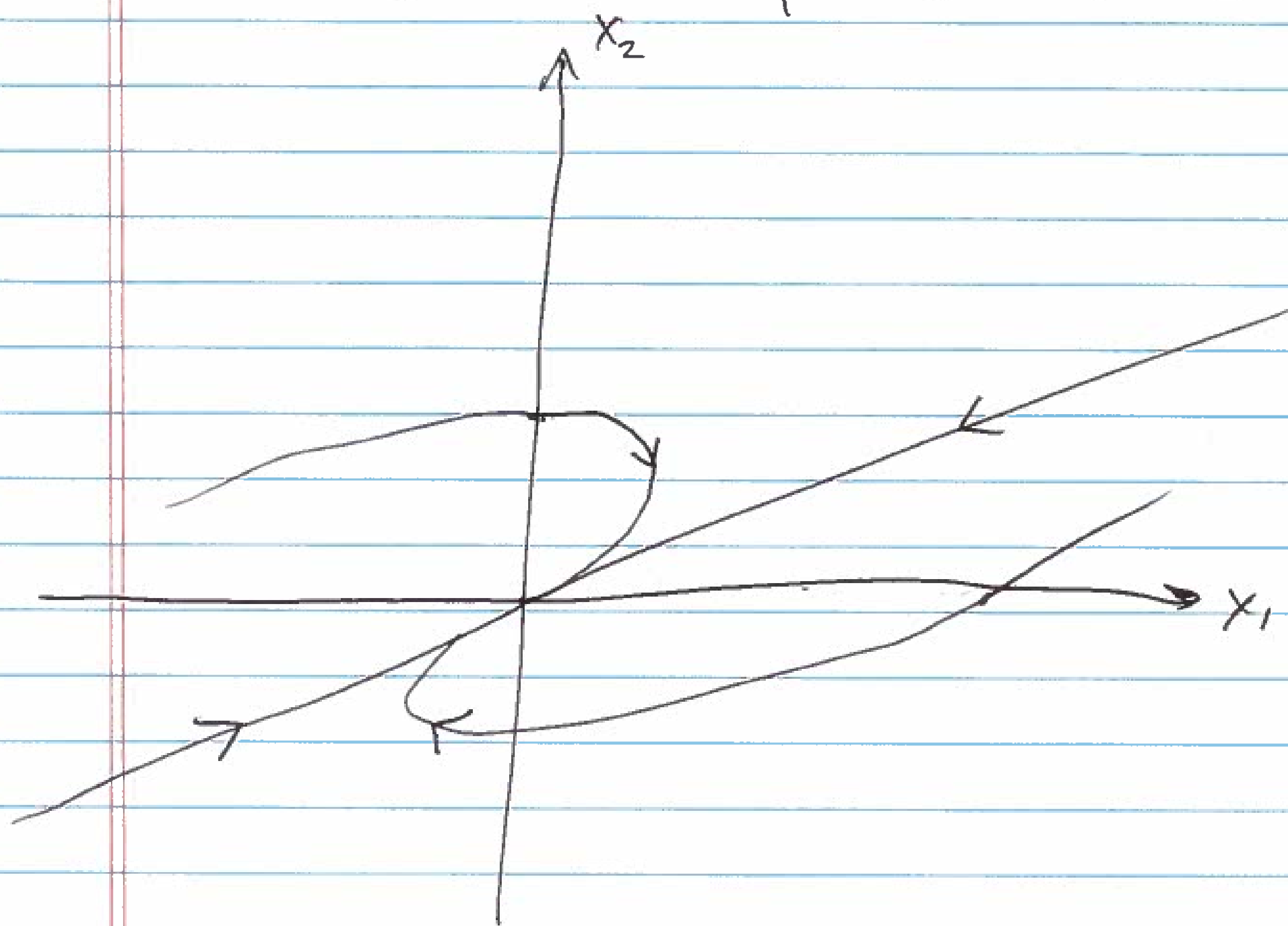
$$(A - rI)\eta = z$$

$$\Rightarrow \begin{pmatrix} -1/2 & 1 \\ -1/4 & 1/2 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\eta_2 = 2 + 1/2\,\eta_1$$

$$\rightarrow \eta = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \frac{\eta_1}{2}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$ part of z already

$$x = c_1 \xi e^{-t} + c_2 \left(\xi t e^{-t} + \eta e^{-t} \right)$$



Direction of trajectories at points:

$$\dot{x} = A \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1/2 \end{pmatrix} = \searrow$$

$$\dot{x} = A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3/2 \\ -1/4 \end{pmatrix} = \swarrow$$