

Homework 3 Solutions

1)

$$y'' + 2y' - 3y = 0$$

$$r^2 + 2r - 3 = 0$$

$$(r+3)(r-1) = 0$$

$$y = c_1 e^{-3x} + c_2 e^x$$

2)

$$y'' + 5y' = 0$$

$$r^2 + 5r = 0$$

$$r(r+5) = 0 \Rightarrow r_1 = 0, r_2 = -5$$

$$y = c_1 + c_2 e^{-5x}$$

3)

$$y'' - 2y' - 2y = 0$$

$$r^2 - 2r - 2 = 0$$

$$r_{1,2} = \frac{2 \pm \sqrt{4+8}}{2} = 1 \pm \sqrt{3}$$

$$y = c_1 e^{(1+\sqrt{3})x} + c_2 e^{(1-\sqrt{3})x}$$

4)

$$y'' + 8y' - 9y = 0 \quad y(1) = 1, y'(1) = 0$$

$$r^2 + 8r - 9 = 0$$

$$(r+9)(r-1) = 0$$

$$y = c_1 e^{-9x} + c_2 e^x$$

$$y(0) = c_1 + c_2 = 1$$

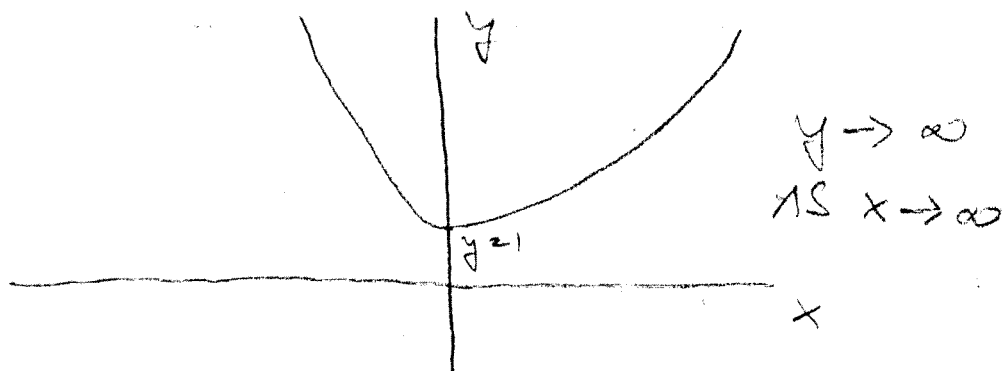
$$y'(0) = -9c_1 + c_2 = 0$$

$$\Rightarrow c_2 = 9c_1$$

$$10c_1 = 1 \Rightarrow c_1 = \frac{1}{10}$$

$$c_2 = \frac{9}{10}$$

$$y = \frac{1}{10} e^{-9x} + \frac{9}{10} e^x$$



5)

$$y'' - 2y' + 6y = 0$$

$$r^2 - 2r + 6 = 0$$

$$r_1, r_2 = \frac{2 \pm \sqrt{4 - 24}}{2} = 1 \pm i\sqrt{5}$$

$$y = c_1 e^x \cos \sqrt{5}x + c_2 e^x \sin \sqrt{5}x$$

(2)

6)

$$y'' + 2y' + 2y = 0$$

$$r^2 + 2r + 2 = 0$$

$$r_{1,2} = \frac{-2 \pm \sqrt{4 - 8}}{2} = -1 \pm i$$

$$y = c_1 e^{-x} \cos x + c_2 e^{-x} \sin x$$

7)

$$y'' + 6y' + 13y = 0$$

$$r^2 + 6r + 13 = 0$$

$$r_{1,2} = \frac{-6 \pm \sqrt{36 - 52}}{2} = -3 \pm i \frac{\sqrt{16}}{2} =$$

$$= -3 \pm 2i$$

$$y = c_1 e^{-3x} \cos 2x + c_2 e^{-3x} \sin 2x$$

8)

$$y'' + 4y' + 5y = 0 \quad y(0) = 1, y'(0) = 0$$

$$r^2 + 4r + 5 = 0$$

$$r_{1,2} = \frac{-4 \pm \sqrt{16 - 20}}{2} = -2 \pm i$$

$$y = c_1 e^{-2x} \cos x + c_2 e^{-2x} \sin x$$

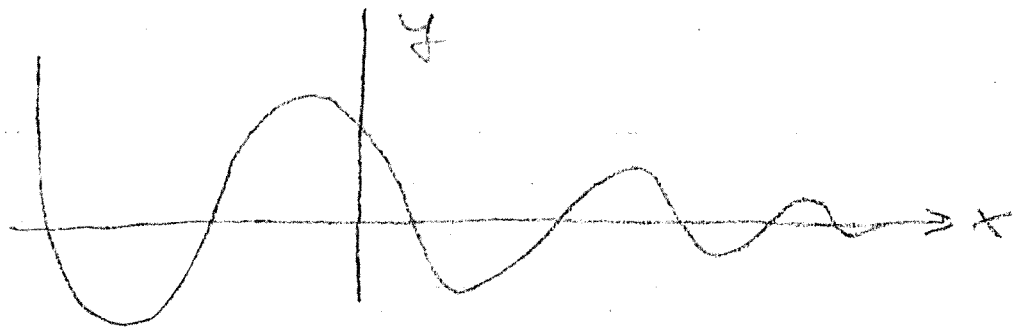
$$y(0) = c_1 = 1$$

$$y'(x) = -2c_1 e^{-2x} \cos x - c_1 e^{-2x} \sin x \\ - 2c_2 e^{-2x} \sin x + c_2 e^{-2x} \cos x$$

$$y'(0) = -2c_1 + c_2 = 0$$

$$c_2 = 2c_1 = 2$$

$$y = e^{-2x} \cos x + 2e^{-2x} \sin x$$



9)

$$y'' - 2y' + y = 0$$

$$r^2 - 2r + 1 = 0$$

$$(r-1)^2 = 0 \quad r_1 = r_2 = 1$$

$$y = c_1 e^x + c_2 x e^x$$

$$10) \quad y'' - 6y' + 9y = 0$$

$$r^2 - 6r + 9 = 0$$

$$(r-3)^2 = 0$$

$$\Rightarrow y = c_1 e^{3x} + c_2 x e^{3x}$$

$$11) \quad y'' + y' - 2y = 2x$$

Complementary solution (homogeneous)

$$y_c'' + y_c' - 2y_c = 0 \quad (*)$$

$$r^2 + r - 2 = 0$$

$$(r+2)(r-1) = 0 \Rightarrow y_c = c_1 e^{-2x} + c_2 e^x$$

$2x$ is a first degree polynomial

$e^{0x} = 1$ is not a solution of $(*)$

$$\Rightarrow \text{Try } y_p = Ax + B$$

$$y_p' = A \quad y_p'' = 0$$

$$\Rightarrow y_p'' + y_p' - 2y_p = A - 2Ax - 2B = 2x$$

$$\Rightarrow (-2A - 2)x + (A - 2B) = 0$$

$$\text{Like terms } \Rightarrow \begin{aligned} -2A - 2 &= 0 \\ A - 2B &= 0 \end{aligned}$$

$$A = -1, \quad B = -\frac{1}{2}$$

$$y_p = -x - \frac{1}{2}$$

$$y = c_1 e^{-2x} + c_2 e^x - x - 1/2$$

$$y(0) = c_1 + c_2 - 1/2 = 0$$

$$y'(0) = -2c_1 + c_2 - 1 = 1$$

$$c_1 + c_2 = 1/2$$

$$2c_1 - c_2 = -2$$

$$\Rightarrow 3c_1 = -3/2$$

$$c_1 = -1/2$$

$$c_2 = 1$$

$$y = e^{-2x} - \frac{1}{2}e^x - x - 1/2$$

$$12) \quad y'' + 4y = x^2 + 3e^x$$

$$y_c'' + 4y_c = 0$$

$$r^2 + 4 = 0 \quad \Rightarrow \quad r = \pm\sqrt{-4} = \pm 2i$$

$$y_c = c_1 \cos 2x + c_2 \sin 2x$$

x^2 - Polynomial of degree 2

e^{0x} - not a sol'n of (*)

e^x - not a sol'n of (*)

$$\text{Try } y_p = Ax^2 + Bx + C + De^x$$

$$y_p' = 2Ax + B + De^x$$

$$y_p'' = 2A + De^x$$

$$y_p'' + 4y_p = 2A + De^x + 4(Ax^2 + Bx + C + De^x)$$

$$= (2A + 4C) + 5De^x$$

$$+ 4Ax^2 + 4Bx$$

$$= x^2 + 3e^x \quad (\text{RHS})$$

Match like terms

$$2A + 4C = 0$$

$$5D = 0$$

$$4A = 1$$

$$4B = 0$$

$$\Rightarrow A = 1/4, B = 0, C = -1/8, D = 3/5$$

$$y = C_1 \cos 2x + C_2 \sin 2x$$

$$+ \frac{x^2}{4} - \frac{1}{8} + \frac{3}{5} e^x$$

$$y(0) = C_1 - \frac{1}{8} + \frac{3}{5} = 0$$

$$y'(0) = 2C_2 + \frac{3}{5} = 2$$

$$\Rightarrow 2C_2 = \frac{7}{5} \Rightarrow C_2 = 7/10$$

$$C_1 = \frac{1}{8} - \frac{3}{5} = \frac{5-24}{8 \cdot 5} = -\frac{19}{40}$$

$$y = -\frac{19}{40} \cos 2x + \frac{7}{10} \sin 2x + \frac{x^2}{4} + \frac{3}{5} e^x - \frac{1}{8}$$

$$13) \quad y'' - 2y' + y = xe^x + 4$$

$$y_c'' - 2y_c' + y_c = 0 \quad (*)$$

$$r^2 - 2r + 1 = 0$$

$$(r-1)^2 = 0 \Rightarrow r_1 = r_2 = 1$$

$$y_c = C_1 e^x + C_2 e^x$$

x - polynomial of degree 1

e^x - does solve $(*)$, so does $x e^x$ (repeated root)

$$\Rightarrow y_{p1} = x^2 (Ax + B) e^x = (Ax^3 + Bx^2) e^x$$

4 - polynomial of degree 0,

$1 = e^0$ does not solve $(*)$

$$y_{p2} = C$$

$$\begin{aligned}
 Y_{p1}' &= (3Ax^2 + 2Bx + Ax^3 + Bx^2)e^x \\
 &= (Ax^3 + (B+3A)x^2 + 2Bx)e^x
 \end{aligned}$$

$$\begin{aligned}
 Y_{p1}'' &= (3Ax^2 + 2(B+3A)x + 2B \\
 &\quad + Ax^3 + (B+3A)x^2 + 2Bx)e^x
 \end{aligned}$$

$$= (Ax^3 + (B+6A)x^2 + (4B+6A)x + 2B)e^x$$

$$\Rightarrow Y_{p1}'' - 2Y_{p1}' + Y_{p1} \quad \downarrow \text{RHS}$$

$$= (4B+6A+4B)x e^x + 2B e^x = x e^x$$

$$\Rightarrow B=0, \quad 6A=1 \Rightarrow A=1/6$$

(match like terms)

Now $Y_{p2} = C, \quad Y_{p2}' = Y_{p2}'' = 0$

$$\Rightarrow Y_{p2}'' - 2Y_{p2}' + Y_{p2} = C = 4$$

$$\Rightarrow C = 4$$

$$\Rightarrow Y_p = Y_{p1} + Y_{p2} = \frac{1}{6}x^3 e^x + 4$$

14)

$$y'' + 3y' = 2x^4 + x^2 e^{-3x} + \sin 3x$$

$$y_c'' + 3y_c' = 0 \quad (*)$$

$$r^2 + 3r = 0 \quad r_1 = 0, r_2 = -3$$

$$y_c = c_1 + c_2 e^{-3x}$$

$2x^4$ - POLYNOMIAL, DEGREE 4.

$e^{0x} = 1$ SOLVES (*), $x e^{-ax} = x$ DOES NOT.

\Rightarrow TRY $y_{p1} = x(Ax^4 + Bx^3 + Cx^2 + Dx + E)$

x^2 - POLYNOMIAL, DEGREE 2.

e^{-3x} SOLVES (*), $x e^{-3x}$ DOES NOT

\Rightarrow TRY $y_{p2} = x(Fx^2 + Gx + H) e^{-3x}$

$\sin 3x$ - DOES NOT SOLVE (*)

TRY $y_{p3} = L \sin 3x + M \cos 3x$

15)

$$y'' + 2y' + 2y = 3e^{-x} + 2e^{-x} \cos x + 4e^{-x} x^2 \sin x$$

$$y_c'' + 2y_c' + 2y_c = 0 \quad (*)$$

$$r^2 + 2r + 2 = 0$$

$$r_1, r_2 = \frac{-2 \pm \sqrt{4 - 8}}{2} = -1 \pm i$$

$$y_c = c_1 e^{-x} \cos x + c_2 e^{-x} \sin x$$

e^{-x} - DOES NOT SOLVE (*)

$$\Rightarrow \text{TRY } y_{p1} = A e^{-x}$$

$e^{-x} \cos x$ AND $e^{-x} \sin x$ SOLVE (*)

x^2 - POLYNOMIAL, ORDER 2

\Rightarrow TRY

$$y_{p2} = x \left[(Bx^2 + Cx + D) e^{-x} \cos x + (Ex^2 + Fx + G) e^{-x} \sin x \right]$$

16)

$$y'' - 4y' + 4y = 2x^2 + 4xe^{2x} + x \sin 2x$$

$$y_c'' - 4y_c' + 4y_c = 0 \quad (*)$$

$$r^2 - 4r + 4 = 0 \Rightarrow (r-2)^2 = 0$$

$$r_1 = r_2 = 2$$

$$y_c = c_1 e^{2x} + c_2 x e^{2x}$$

$2x^2$ - POLYNOMIAL, ORDER 2

$$\text{TRY } y_{p1} = Ax^2 + Bx + C$$

x - POLYNOMIAL, ORDER 1

e^{2x} - SOLVES (*), SO DOES $x e^{2x}$

$$\Rightarrow \text{TRY } y_{p2} = x^2 (Dx + E) e^{2x}$$

$\sin 2x$ - DOES NOT SOLVE (*)

x - POLYNOMIAL, ORDER 1

$$\Rightarrow \text{TRY } y_{p3} = (Fx + G) \sin 2x + (Hx + I) \cos 2x$$

17)

$$y'' + 2y' = 3 + 4 \sin 2x$$

$$y_c'' + 2y_c' = 0 \quad (*)$$

$$r^2 + 2r = 0 \Rightarrow r(r+2) = 0$$

$$r_1 = 0, r_2 = -2$$

$$y_c = c_1 + c_2 e^{-2x}$$

$$y_{p1}'' + 2y_{p1}' = 3$$

$$3 = 3 \cdot 1; \quad 1 \text{ solves } (*)$$

$x \cdot 1 = x$ DOES NOT SOLVE (*)

$$\Rightarrow y_{p1} = Ax, \quad y_{p1}' = A, \quad y_{p1}'' = 0$$

$$2A = 3 \Rightarrow A = \frac{3}{2}$$

$$y_{p2}'' + 2y_{p2}' = 2 \sin 2x$$

$\sin 2x$ - DOES NOT SOLVE (*)

$$y_{p2} = A \sin 2x + B \cos 2x$$

$$y_{p2}' = 2A \cos 2x - 2B \sin 2x$$

$$y_{p2}'' = -4A \sin 2x - 4B \cos 2x$$

$$-4A \sin 2x - 4B \cos 2x$$

$$+ 4A \cos 2x - 4B \sin 2x = 4 \sin 2x$$

$$-A - B = 4 \quad \Rightarrow \quad -2A = 4$$

$$A - B = 0 \quad A = -\frac{1}{2} = B$$

$$y_p = -\frac{1}{2} (\sin 2x + \cos 2x)$$

$$y = c_1 + c_2 e^{-2x} + \frac{3}{2} - \frac{1}{2} (\sin 2x + \cos 2x)$$

18)

$$2y'' + 3y' + y = x^2 + 3 \sin x$$

$$2y_c'' + 3y_c' + y_c = 0 \quad (*)$$

$$2r^2 + 3r + 1 = 0$$

$$(2r+1)(r+1) = 0$$

$$r_1 = -\frac{1}{2}, \quad r_2 = -1$$

$$y_c = c_1 e^{-\frac{x}{2}} + c_2 e^{-x}$$

$$2y'' + 3y' + y = x^2$$

$x^2 =$ QUADRATIC $\cdot 1$

1 - DOES NOT SOLVE (*)

TRY $y_{p1} = Ax^2 + Bx + C$

$$y'_{p1} = 2Ax + B$$

$$y''_{p1} = 2A$$

$$4A + 6Ax + 3B + Ax^2 + Bx + C = x^2$$

$$A = 1$$

$$6A + B = 0 \Rightarrow B = -6$$

$$4A + 3B + C = 0 \Rightarrow 4 - 18 + C = 0$$

$$C = 14$$

$$y_{p1} = x^2 - 6x + 14$$

$$2y'' + 3y' + y = 3 \cos x$$

Ans x - DOES NOT SOLVE *

$$\text{Try } y_{p2} = A \sin x + B \cos x$$

$$y_{p2}' = A \cos x - B \sin x$$

$$y_{p2}'' = -A \sin x - B \cos x$$

$$-2A \sin x - 2B \cos x + 3A \cos x$$

$$- 3B \sin x + A \sin x + B \cos x = 3 \sin x$$

$$(-2A - 3B + A) = 3$$

$$(-2B + 3A + B) = 0$$

$$-A - 3B = 3$$

$$3A - B = 0 \quad B = 3A$$

$$-10A = 3 \Rightarrow A = -\frac{3}{10}$$

$$B = -\frac{9}{10}$$

$$y_{p2} = -\frac{3}{10} \sin x - \frac{9}{10} \cos x$$

$$y = c_1 e^{-\frac{x}{2}} + c_2 e^{-x} + x^2 - 6x + 14$$

$$- \frac{3}{10} \sin x - \frac{9}{10} \cos x$$

-19)

$$\ddot{u} + \omega_0^2 u = r \cos \omega t$$

$$\ddot{u}_c + \omega_0^2 u_c = 0 \quad (*)$$

$$r^2 + \omega_0^2 = 0, \quad r = \pm i \omega_0$$

$$u_c = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$$

$$\ddot{u}_p + \omega_0^2 u_p = r \cos \omega t$$

$$r \cos \omega t \text{ solves } (*)$$

$$\text{TRY } u_p = t (A \cos \omega t + B \sin \omega t)$$

$$\dot{u}_p = A \cos \omega t + B \sin \omega t +$$

$$t \omega (-A \sin \omega t + B \cos \omega t)$$

$$\ddot{u}_p = -\omega^2 t (A \cos \omega t + B \sin \omega t)$$

$$+ 2\omega_0 (-A \sin \omega t + B \cos \omega t)$$

$$\begin{aligned} \ddot{u}_p + \omega_0^2 u_p &= 2\omega_0 B \cos \omega t - 2\omega_0 A \sin \omega t \\ &= r \cos \omega t \end{aligned}$$

$$B = \frac{1}{2\omega_0}, \quad A = 0$$

$$u_p = \frac{t}{2\omega_0} \sin \omega t$$

$$u = C_1 \cos \omega t + C_2 \sin \omega t + \frac{1}{2\omega_0} t \sin \omega t$$

20)

$$y'' + y = x(1 + \sin x)$$

$$y_c'' + y_c = 0 \quad (*)$$

$$y_c = C_1 \cos x + C_2 \sin x$$

$$y_{p1}'' + y_{p1} = x$$

$x = x \cdot 1$, 1 DOES NOT SOLVE (*)

TRY $y_{p1} = Ax + B$

$$y_{p2}'' + y_{p2} = x \sin x$$

$x =$ LINEAR POLYNOMIAL

$\sin x$ - SOLVES (*)

TRY

$$y_{p2} = x((Cx + D) \sin x + (Ex + F) \cos x)$$

(21)

$$y'' - 5y' + 6y = e^x \cos 2x + e^{2x} (3x+4) \sin x$$

$$y_c'' - 5y_c' + 6y_c = 0$$

$$r^2 - 5r + 6 = 0$$

$$(r-2)(r-3) = 0$$

$$r_1 = 2, r_2 = 3$$

$$y = c_1 e^{2x} + c_2 e^{3x}$$

$$y_{p1}'' - 5y_{p1}' + 6y_{p1} = e^x \cos 2x$$

$e^x \cos 2x$ DOES NOT SOLVE (*)

$$\Rightarrow y_{p1} = A e^x \cos 2x + B e^x \sin 2x$$

$$y_{p2}'' - 5y_{p2}' + 6y_{p2} = e^{2x} (3x+4) \sin x$$

$e^{2x} \sin x$ - DOES NOT SOLVE (*)

$3x+4$ - LINEAR POLYNOMIAL

$$\Rightarrow y_{p2} = (Cx+D) e^{2x} \sin x + (Ex+F) e^{2x} \cos x$$