

Homework 2 Solutions

MATH 2400

①

$$1) \quad \frac{dy}{dx} = y' = \frac{x^2}{y} \quad y \neq 0$$

$$y \, dy = x^2 \, dx$$

$$\frac{1}{2} y^2 = \frac{1}{3} x^3 + C$$

$$y = \pm \sqrt{\frac{2}{3} x^3 + C}$$

$$2) \quad y' + y^2 \sin x = 0$$

$$\frac{dy}{dx} = -y^2 \sin x$$

$$\frac{1}{y^2} \, dy = -\sin x \, dx \quad \left(\begin{array}{l} \text{IF} \\ y \neq 0 \end{array} \right)$$

$$-\frac{1}{y} = \cos x + C$$

$$y = \frac{-1}{\cos x + C}$$

Also by inspection, $y = 0$ is a solution

$$3) \quad xy' = (1-y^2)^{1/2}$$

$$\frac{dy}{dx} = \frac{(1-y^2)^{1/2}}{x}$$

$$\Rightarrow \frac{dy}{(1-y^2)^{1/2}} = \frac{dx}{x} \quad (-1 \leq y \leq 1)$$

$$\Rightarrow \arcsin y = \log |x| + C, \quad x \neq 0$$

$$\Rightarrow y = \sin(\log |x| + C)$$

Also $y = \pm 1$ is a solution by inspection

$$4) \quad y' = \frac{x(x^2+1)}{4y^3} \quad y(0) = -\frac{1}{\sqrt{2}}$$

$$4y^3 dy = (x^3 + x) dx$$

$$y^4 = \frac{x^4}{4} + \frac{x^2}{2} + C$$

$$y = \pm \left(\frac{x^4}{4} + \frac{x^2}{2} + C \right)^{1/4}$$

$$y(0) = -\frac{1}{\sqrt{2}} = -C^{1/4}$$

$$\Rightarrow C = \frac{1}{4}$$

$$\Rightarrow y = - \left(\frac{x^4}{4} + \frac{x^2}{2} + \frac{1}{4} \right)^{1/4}$$

$$= - \left(\frac{x^4 + 2x^2 + 1}{4} \right)^{1/4}$$

$$= - \left(\frac{(x^2+1)^2}{2^2} \right)^{1/4} = - \sqrt{\frac{(x^2+1)}{2}}$$

defined on $-\infty < x < \infty$

$$5) \quad y' = \frac{x^2}{y(1+x^3)} \quad y(0) = -1$$

$$y \, dy = \frac{x^2 \, dx}{1+x^3}$$

$$\frac{1}{2} y^2 = \frac{1}{3} \log |1+x^3| + C$$

$$y = \pm \sqrt{\frac{2}{3} \log |1+x^3| + 2C}$$

$$y(0) = -1 = \pm \sqrt{\frac{2}{3} \log 1 + 2C} = \pm \sqrt{2C}$$

Choose $-\sqrt{2C}$

$$\Rightarrow 2C = 1$$

$$y = - \sqrt{\frac{2}{3} \log |1+x^3| + 1}$$

$$6) \frac{dQ}{dt} = -rQ$$

$$\Rightarrow Q = Q_0 e^{-rt}$$

lose $\frac{1}{3}$ of mass when $T = 11.7$

$$\Rightarrow Q(T) = \frac{2}{3} \quad \text{when} \quad Q(0) = Q_0 = 1$$

$$\Rightarrow \frac{2}{3} = e^{-rT}$$

$$\Rightarrow r = \frac{1}{T} \log \frac{3}{2}$$

Half life

$$Q(\tau) = \frac{1}{2} \quad \text{when} \quad Q(0) = Q_0 = 1$$

$$\Rightarrow \frac{1}{2} = e^{-r\tau}$$

$$\begin{aligned} \Rightarrow \tau &= \frac{1}{r} \log 2 = \frac{T \log 2}{\log \frac{3}{2}} \\ &= \frac{11.7 \log 2}{\log \frac{3}{2}} \approx 20.0 \text{ days} \end{aligned}$$

(5)

$$7) \quad \frac{dQ}{dt} = -rQ + k$$

$$r = 0.02828 \text{ days}^{-1}$$

$$k = 1 \text{ mg/day}$$

$$(a) \quad \frac{dQ}{dt} + rQ = k$$

$$\Rightarrow u(t) = e^{rt}$$

$$\frac{d}{dt} [e^{rt} Q] = e^{rt} k$$

$$e^{rt} Q = \frac{k}{r} e^{rt} + C$$

$$Q(t) = \frac{k}{r} + C e^{-rt}$$

$$Q(0) = Q_0 = \frac{k}{r} + C$$

$$C = Q_0 - \frac{k}{r}$$

$$Q(t) = \frac{k}{r} (Q_0 - \frac{k}{r}) e^{-rt}$$

$$= Q_0 e^{-rt} + \frac{k}{r} (1 - e^{-rt})$$

(6)

$$Q_0 = 100 \text{ mg}$$

$$\frac{k}{r} \approx 35.36 \quad \left(Q_0 - \frac{k}{r}\right) \approx 64.64$$

$$Q(t) = 35.36 + 64.64 e^{-0.02828t}$$

$$(b) \quad \lim_{t \rightarrow \infty} Q(t) = \frac{k}{r} = 35.36 \text{ mg} = Q_1$$

$$(c) \quad Q = 35.85 \text{ mg} = Q_1$$

$$Q(T) = \frac{k}{r} + \left(Q_0 - \frac{k}{r}\right) e^{-rT}$$

$$e^{rT} = \frac{Q(T) - \frac{k}{r}}{Q_0 - \frac{k}{r}}$$

$$T = \frac{1}{r} \ln \frac{Q_0 - \frac{k}{r}}{Q(T) - \frac{k}{r}}$$

$$\approx 171.9 \text{ days}$$

$$(d) \quad \frac{dQ}{dt} = 0 \text{ for constant level}$$

$$\Rightarrow \frac{dQ}{dt} + rQ = rQ = k = (100 \text{ mg}) \times r$$

$$= 2.828 \text{ mg}$$

$$8) \quad V = 120 \text{ lt.} \quad \phi = 2 \text{ lt./min.}$$

$$\frac{dQ}{dt} = \phi (Y - Q/V)$$

In class we got

$$Q(t) = VY + (Q_0 - VY) e^{-\phi t/V}$$

Initially pure water $\Rightarrow Q_0 = 0$

$$\begin{aligned} Q(t) &= VY (1 - e^{-\phi t/V}) \\ &= 120Y (1 - e^{-t/60}) \end{aligned}$$

$$\lim_{t \rightarrow \infty} Q(t) = 120Y$$

↑ limiting amount

$$9) \quad V = 1200 \text{ ft}^3$$

$$C = 0.04$$

$$\phi = 0.1 \text{ ft}^3/\text{min.}$$

From class,

$$Q(t) = VC + (Q_0 - VC) e^{-\phi t/V}$$

↑ Quantity of Carbon monoxide

Concentration

$$X(t) = \frac{Q(t)}{V} = C + \left(\frac{Q_0}{V} - C \right) e^{-\phi t/V}$$

Initially no smoke $\Rightarrow Q_0 = 0$

$$\begin{aligned} X(t) &= C (1 - e^{-\phi t/V}) \\ &= 0.04 (1 - e^{-t/12000}) \end{aligned}$$

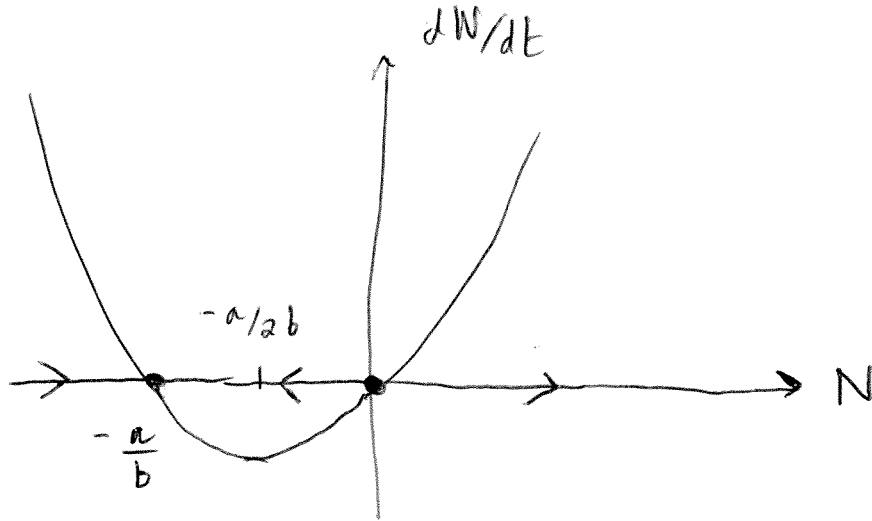
$$\begin{aligned} b) \quad X(\tau) &= 0.00012 \\ &= C (1 - e^{-\phi \tau/V}) \end{aligned}$$

$$\Rightarrow C e^{-\phi \tau/V} = C - X(\tau)$$

$$e^{-\phi \tau/V} = 1 - \frac{X(\tau)}{C}$$

$$\tau = \frac{V}{\phi} \log \frac{1}{1 - \frac{X(\tau)}{C}} = 36 \text{ min}$$

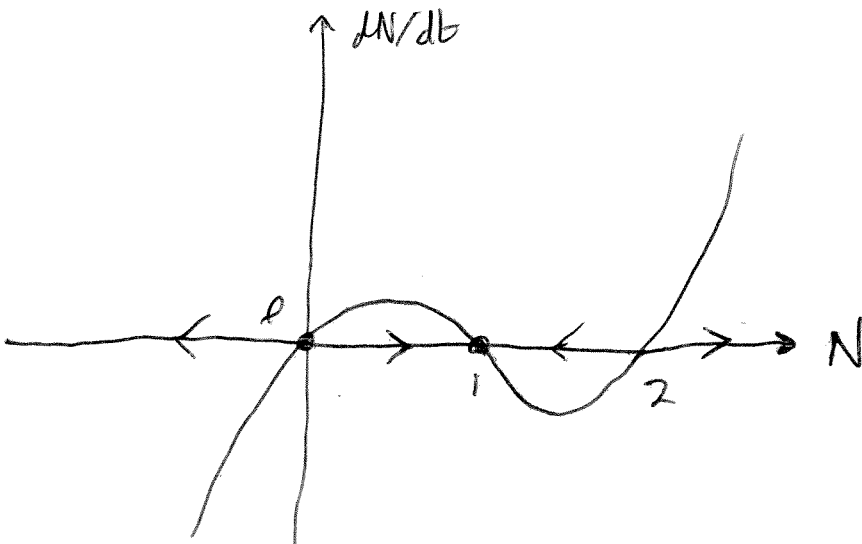
10) $\frac{dN}{dt} = aN + bN^2, \quad a, b > 0$



Equilibria
 $N = 0$ unstable
 $N = -\frac{a}{b}$ stable

11) $\frac{dN}{dt} = N(N-1)(N-2)$

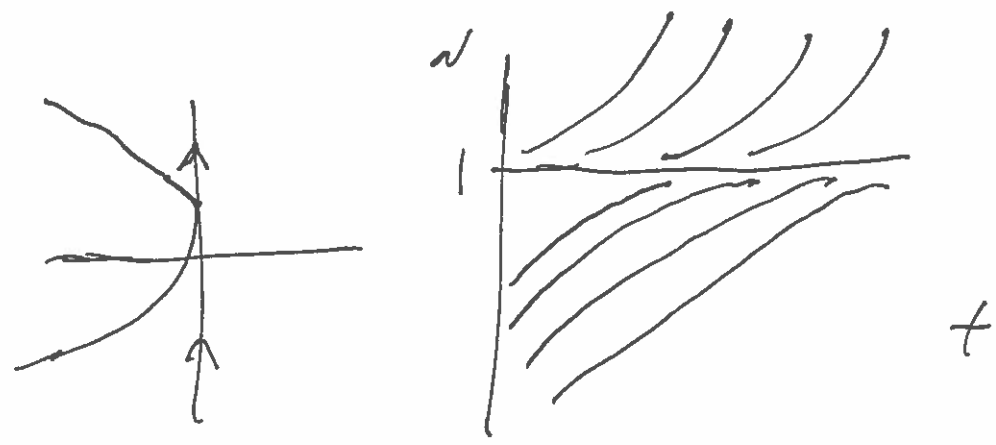
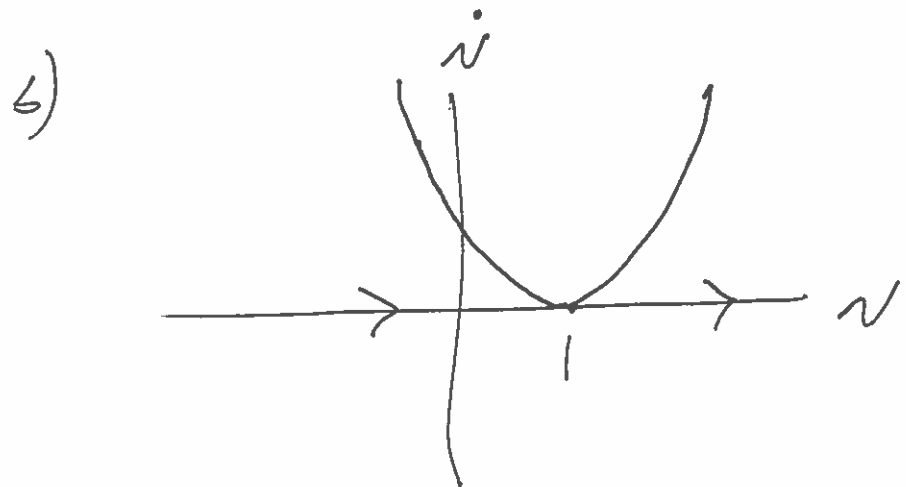
Equilibria
 $N = 0$ unstable
 $N = 1$ stable
 $N = 2$ unstable



$$b) \frac{dn}{dt} = k(1-n)^2$$

(10)

a) $k(1-n)^2 = 0 \Rightarrow n=1$ (Double Root) Only Equilibrium.



$$c) \frac{dN}{dt} = k(1-N)^2$$

$$\frac{dN}{(1-N)^2} = k dt \Rightarrow \int \frac{dN}{(1-N)^2} = \int k dt$$

$$\Rightarrow \frac{1}{1-N} = kt + C$$

$$N(0) = N_0 \Rightarrow \frac{1}{1-N_0} = C$$

$$\Rightarrow C = \frac{1}{1-N_0}$$

$$\frac{1}{1-N} = \frac{1}{1-N_0} + kt$$

$$1-N = \frac{1-N_0}{1+(1-N_0)kt}$$

$$N = 1 - \frac{1-N_0}{1+(1-N_0)kt}$$

$$= \frac{N_0 + (1-N_0)kt}{1+(1-N_0)kt}$$

(2)

$$\frac{dN}{dt} = \frac{(1-N_0)^2}{(1+(1-N_0)kt)^2} > 0 \quad \text{for } N_0 \neq 1$$

$$\frac{dN}{dt} = 0 \quad \text{for } N = 1$$

$$N_0 < 1 : \quad \lim_{t \rightarrow \infty} N(t) = 1$$

$$N_0 > 1 : \quad \lim_{t \rightarrow \infty} N(t) = \infty$$

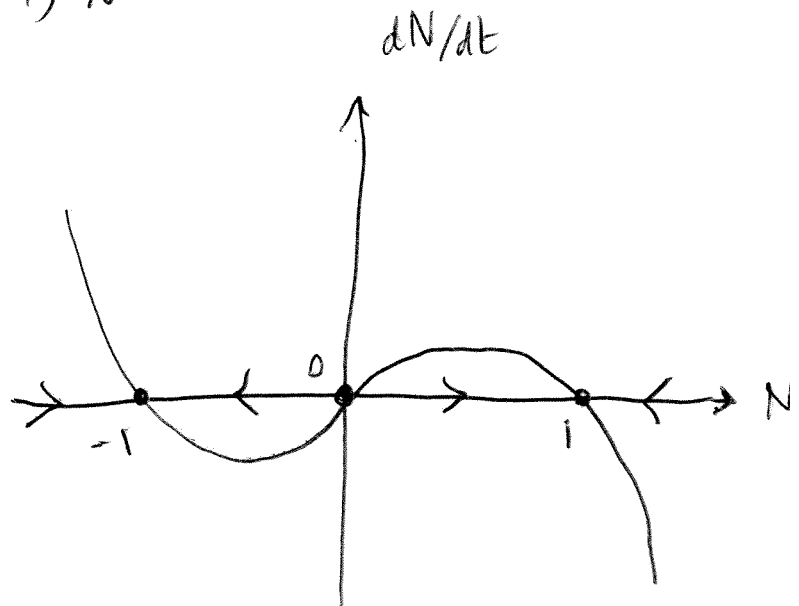
$$(3) \quad \frac{dN}{dt} = N(1-N^2)$$

$$\frac{dN}{dt} = -(N+1)(N-1)N$$

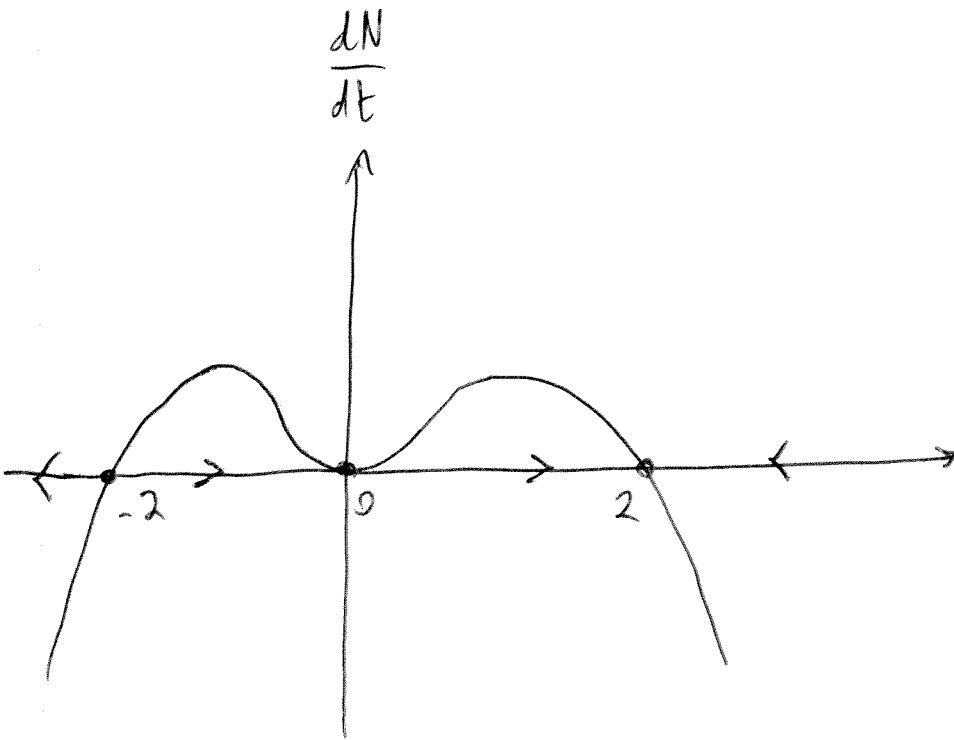
Equilibria

 $N = -1$ stable

 $N = 0$ unstable

 $N = 1$ stable


$$14) \quad \frac{dN}{dt} = N^2 (4 - N^2) = N^2 (2 - N)(2 + N)$$



Equilibria

 $N = -2$ unstable $N = 0$ semistable $N = 2$ unstable

$$15) \quad \frac{dN}{dt} = rN \left(1 - \frac{N}{k}\right) - h$$

(a) Equilibria

$$rN \left(1 - \frac{N}{k}\right) - h = 0$$

$$rN - \frac{r}{k}N^2 - h = 0$$

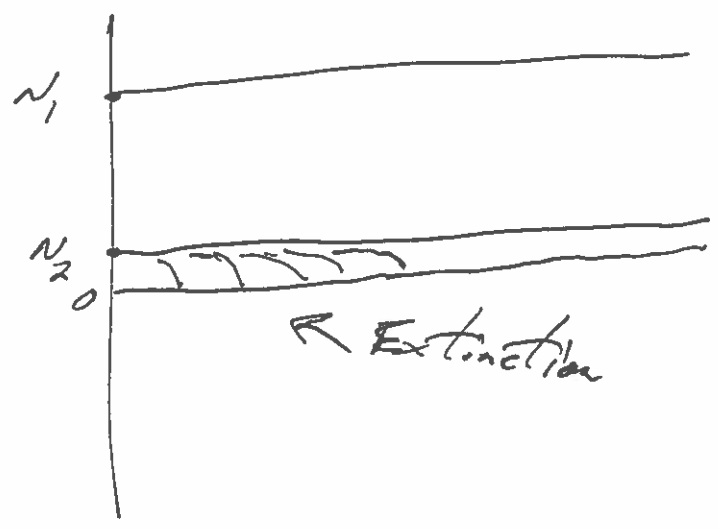
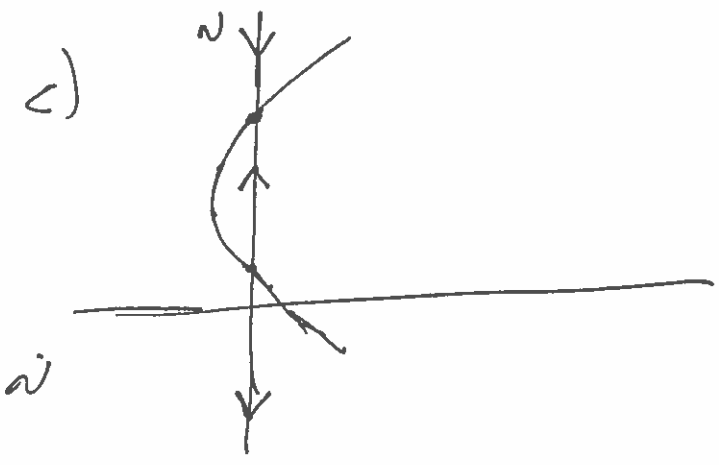
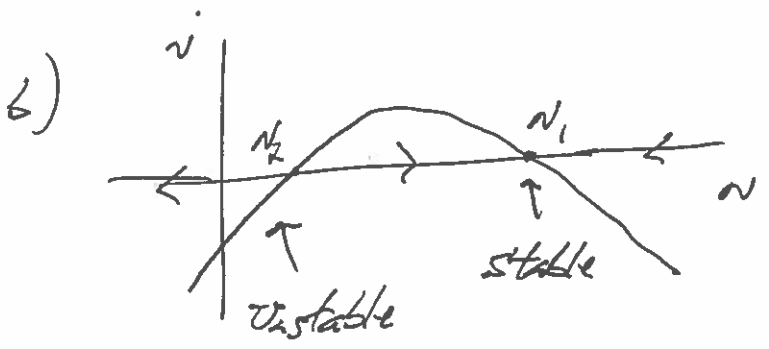
$$N^2 - kN + \frac{kh}{r} = 0$$

$$\Rightarrow N_1, N_2 = \frac{k}{2} \pm \sqrt{\left(\frac{k}{2}\right)^2 - \frac{kh}{r}}$$

For 2 Real Sol's

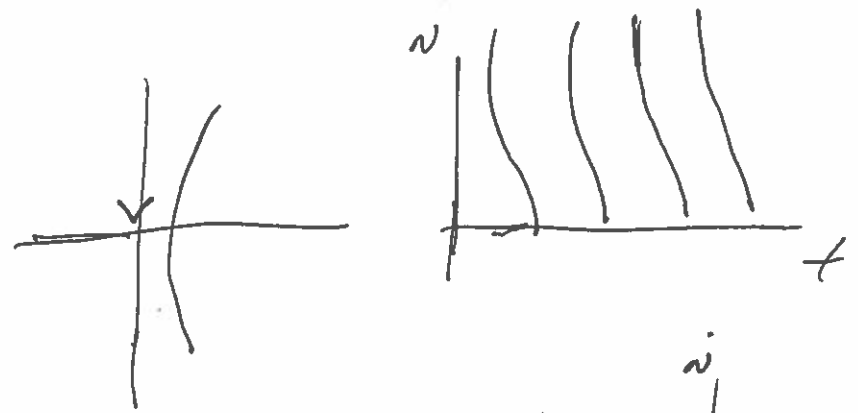
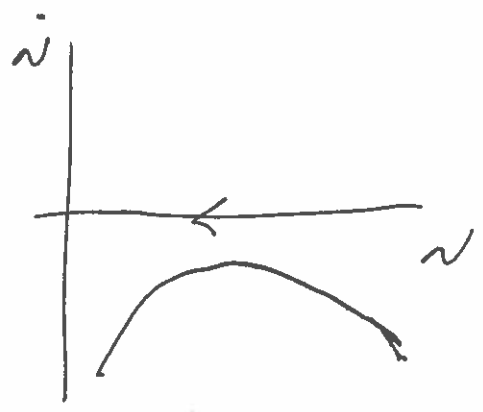
$$\left(\frac{K}{2}\right)^2 - \frac{Kh}{r} > 0. \text{ This occurs when}$$

$$\left(\frac{K}{2}\right)^2 > \frac{Kh}{r} \Rightarrow h < \frac{rK}{4} \Rightarrow \text{Two Real Equilibria.}$$



(c) If $k > 4$, then $(\frac{r}{2}) - \frac{r}{r} < 0$, and solns (and thus

ν_1, ν_2) are complex.



e) $k = \frac{rk}{4} \Rightarrow \nu_1 = \nu_2 = \frac{k}{2}$

