Homework 2 Solutions   MATH 2400

1) $\frac{dy}{dx} = y' = \frac{x^2}{y}$ $y \neq 0$

$y\,dy = x^2 dx$

$\frac{1}{2} y^2 = \frac{1}{3} x^3 + C$

$y = \pm\sqrt{\frac{2}{3} x^3 + C}$

2) $y' + y^2 \sin x = 0$

$\frac{dy}{dx} = -y^2 \sin x$

$\frac{1}{y^2} dy = -\sin x\, dx$ $\left(\text{If } y \neq 0 \right)$

$-\frac{1}{y} = \cos x + C$

$y = \frac{-1}{\cos x + C}$

Also by inspection, $y = 0$ is a solution

3) $xy' = (1-y^2)^{1/2}$

$\frac{dy}{dx} = \frac{(1-y^2)^{1/2}}{x}$

$$\Rightarrow \frac{dy}{(1-y^2)^{1/2}} = \frac{dx}{x} \qquad (-1 \le y \le 1)$$

$$\Rightarrow \arcsin y = \log|x| + c, \qquad x \neq 0$$

$$\Rightarrow y = \sin(\log|x| + c)$$

Also $y = \pm 1$ is a solution by inspection

4) $y' = \dfrac{x(x^2+1)}{4y^3}$ $\qquad y(0) = -1/\sqrt{2}$

$$4y^3\,dy = (x^3 + x)\,dx$$

$$y^4 = \frac{x^4}{4} + \frac{x^2}{2} + C$$

$$y = \pm\left(\frac{x^4}{4} + \frac{x^2}{2} + c\right)^{1/4}$$

$$y(0) = \frac{-1}{\sqrt{2}} = -C^{1/4}$$

$$\Rightarrow c = 1/4$$

$$\Rightarrow y = -\left(\frac{x^4}{4} + \frac{x^2}{2} + \frac{1}{4}\right)^{1/4}$$

$$= -\left(\frac{x^4 + 2x^2 + 1}{4}\right)^{1/4}$$

$$= -\left(\frac{(x^2+1)^2}{2^2}\right)^{1/4} = -\sqrt{\frac{(x^2+1)}{2}}$$

defined on $-\infty < x < \infty$

5) $$y' = \frac{x^2}{y(1+x^3)} \qquad y(0) = -1$$

$$y\,dy = \frac{x^2 dx}{1+x^3}$$

$$\frac{1}{2}y^2 = \frac{1}{3}\log|1+x^3| + C$$

$$y = \pm\sqrt{\frac{2}{3}\log|1+x^3| + 2C}$$

$$y(0) = -1 = \pm\sqrt{\frac{2}{3}\log 1 + 2C} = \pm\sqrt{2C}$$

Choose $-\sqrt{2C}$

$$\Rightarrow 2C = 1$$

$$y = -\sqrt{\frac{2}{3}\log|1+x^3| + 1}$$

6) $\frac{dQ}{dt} = -rQ$

$\Rightarrow Q = Q_0 e^{-rt}$

lose $1/3$ of mass when $T = 11.7$

$\Rightarrow Q(T) = \frac{2}{3}$ when $Q(0) = Q_0 = 1$

$\Rightarrow \frac{2}{3} = e^{-rT}$

$\Rightarrow r = \frac{1}{T} \log \frac{3}{2}$

Half life

$Q(\tau) = \frac{1}{2}$ when $Q(0) = Q_0 = 1$

$\Rightarrow \frac{1}{2} = e^{-r\tau}$

$\Rightarrow \tau = \frac{1}{r} \log 2 = \frac{T \log 2}{\log 3/2}$

$= \frac{11.7 \log 2}{\log 3/2} \approx 20.0$ days

7) $$\frac{dQ}{dt} = -rQ + k$$

$$r = 0.02828 \text{ days}^{-1}$$
$$k = 1 \text{ mg/day}$$

(a) $$\frac{dQ}{dt} + rQ = k$$

$$\Rightarrow \mu(t) = e^{rt}$$

$$\frac{d}{dt}\left[e^{rt}Q\right] = e^{rt}k$$

$$e^{rt}Q = \frac{k}{r}e^{rt} + C$$

$$Q(t) = \frac{k}{r} + Ce^{-rt}$$

$$Q(0) = Q_0 = \frac{k}{r} + C$$

$$C = Q_0 - \frac{k}{r}$$

$$Q(t) = \frac{k}{r} (Q_0 - k/r)\, e^{-rt}$$

$$= Q_0 e^{-rt} + \frac{k}{r}(1 - e^{-rt})$$

$$Q_0 = 100 \text{ mg}$$

$$\frac{k}{r} \simeq 35.36 \qquad \left(Q_0 - \frac{k}{r}\right) \approx 64.64$$

$$Q(t) = 35.36 + 64.64\, e^{-0.02828t}$$

(b) $$\lim_{t \to \infty} Q(t) = \frac{k}{r} = 35.36 \text{ mg} = Q_1$$

(c) $$Q = 35.85 \text{ mg} = Q_1$$

$$Q(T) = \frac{k}{r} + \left(Q_0 - \frac{k}{r}\right) e^{-rT}$$

$$e^{-rT} = \frac{Q(T) - k/r}{Q_0 - k/r}$$

$$T = \frac{1}{r} \ln \frac{Q_0 - k/r}{Q(T) - k/r}$$

$$\approx 171.9 \text{ days}$$

(d) $\frac{dQ}{dt} = 0$ for constant level

$$\Rightarrow \quad \frac{dQ}{dt} + rQ = rQ = k = (100 \text{ mg}) \times r$$

$$= 2.828 \text{ mg}$$

8) $V = 120$ lit. $\quad \phi = 2$ lit./min.

$$\frac{dQ}{dt} = \phi\left(\gamma - Q/V\right)$$

In class we got

$$Q(t) = V\gamma + (Q_0 - V\gamma)e^{-\phi t/V}$$

Initially pure water $\Rightarrow Q_0 = 0$

$$Q(t) = V\gamma\left(1 - e^{-\phi t/V}\right)$$

$$= 120\gamma\left(1 - e^{-t/60}\right)$$

$$\lim_{t \to \infty} Q(t) = 120\gamma$$

↑ limiting amount

9) $V = 1200\ ft^3$

$C = 0.04 \qquad \phi = 0.1\ ft^3/min.$

From class,

$$Q(t) = VC + (Q_0 - VC)e^{-\phi t/V}$$

↑ Quantity of Carbon monoxide

Concentration

$$X(t) = \frac{Q(t)}{V} = C + \left(\frac{Q_0}{V} - C\right) e^{-\phi t/V}$$

Initially no smoke $\Rightarrow$ $Q_0 = 0$

$$X(t) = C\left(1 - e^{-\phi t/V}\right)$$

$$= 0.04\left(1 - e^{-t/12000}\right)$$

b) $X(\tau) = 0.00012$

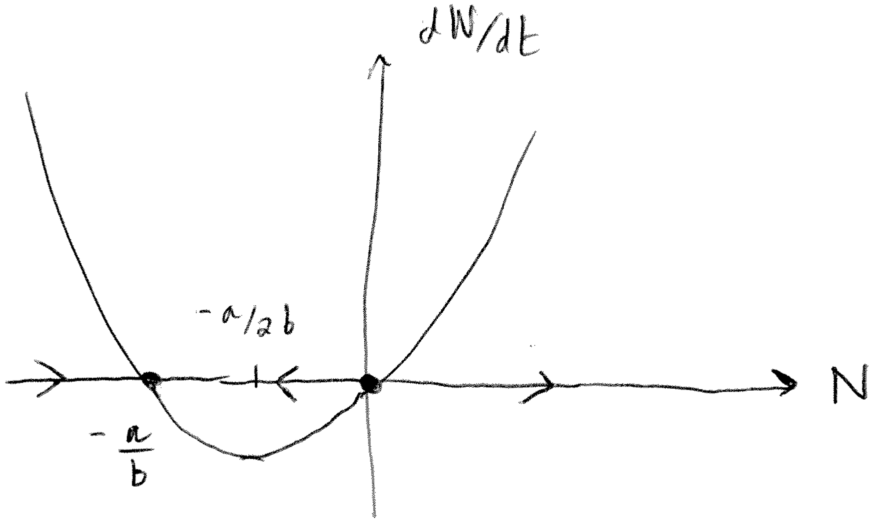
$$= C\left(1 - e^{-\phi \tau/V}\right)$$

$$\Rightarrow \quad C e^{-\phi \tau/V} = C - X(\tau)$$

$$e^{-\phi\tau/V} = 1 - \frac{X(\tau)}{C}$$

$$\tau = \frac{V}{\phi} \log \frac{1}{1 - \frac{X(\tau)}{C}} = 36 \text{ min}$$

10) $\frac{dN}{dt} = aN + bN^2, \qquad a, b > 0$



Equilibria

$N = 0$ unstable

$N = -\frac{a}{b}$ stable

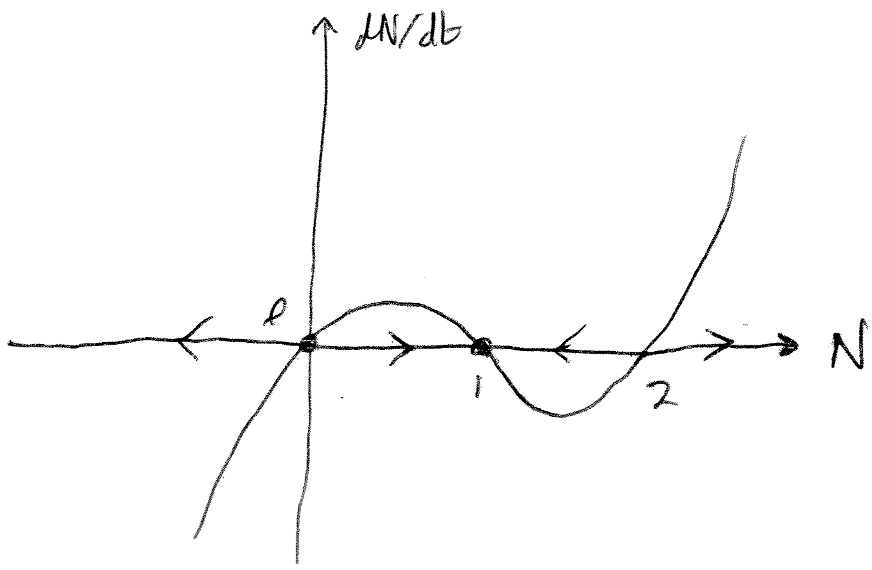
11) $\frac{dN}{dt} = N(N-1)(N-2)$

Equilibria

$N = 0$ unstable

$N = 1$ stable

$N = 2$ unstable



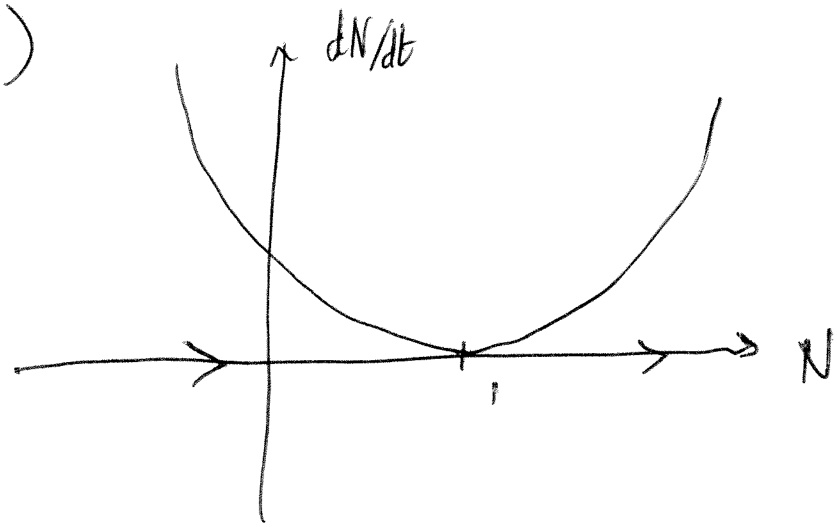
12) $\dfrac{dN}{dt} = k(1-N)^2$

a) $k(1-N)^2 = 0 \Rightarrow 1-N=0$

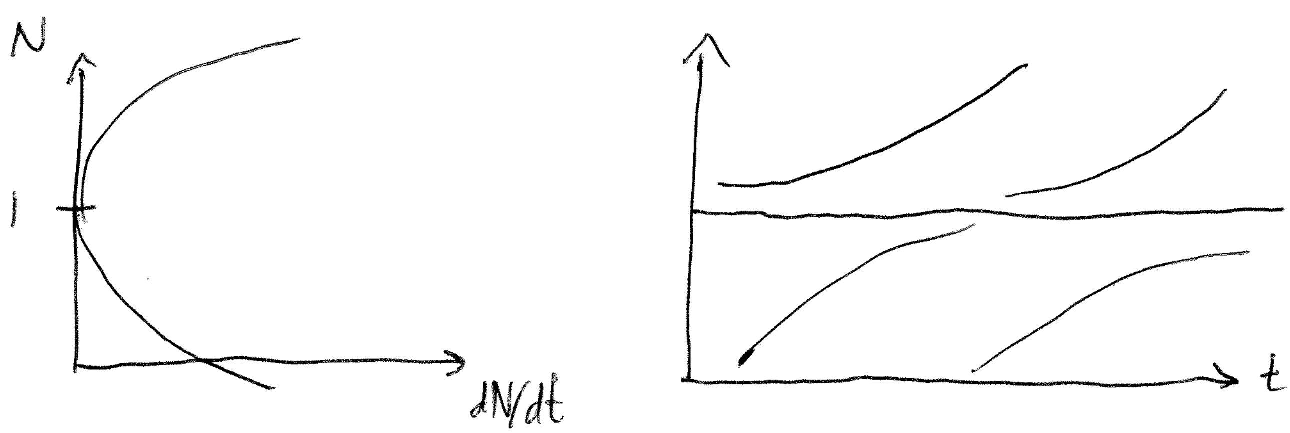
$\Rightarrow N=1$

only equilibrium

b)



$\Rightarrow$



c) $$\frac{dN}{dt} = k(1-N)^2$$

$$\frac{dN}{(1-N)^2} = k\,dt \Rightarrow \int \frac{dN}{(1-N)^2} = \int k\,dt$$

$$\Rightarrow \frac{1}{1-N} = kt + c$$

$$N(0) = N_0 \Rightarrow \frac{1}{1-N_0} = c$$

$$\Rightarrow c = \frac{1}{1-N_0}$$

$$\frac{1}{1-N} = \frac{1}{1-N_0} + kt$$

$$1-N = \frac{1-N_0}{1+(1-N_0)kt}$$

$$N = 1 - \frac{1-N_0}{1+(1-N_0)kt}$$

$$= \frac{N_0 + (1-N_0)kt}{1+(1-N_0)kt}$$

$$\frac{dN}{dt} = \frac{(1-N_0)^2}{(1+(1-N_0)kt)^2} > 0 \qquad \text{for } N_0 \neq 1$$

$$\frac{dN}{dt} = 0 \quad \text{for } N = 1$$

$N_0 < 1$: $\quad \lim_{t \to \infty} N(t) = 1$

$N_0 > 1$: $\quad \lim_{t \to \infty} N(t) = \infty$

13) $$\frac{dN}{dt} = N(1-N^2)$$

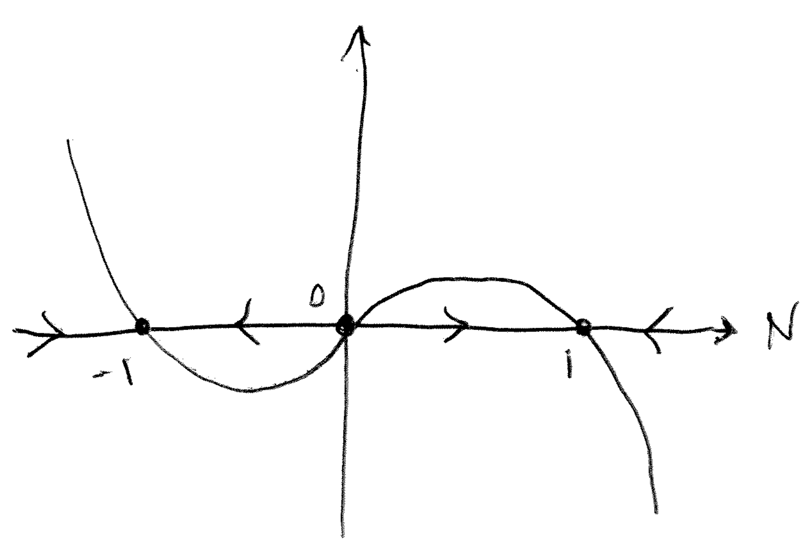
$$\frac{dN}{dt} = -(N+1)(N-1)N$$

Equilibria

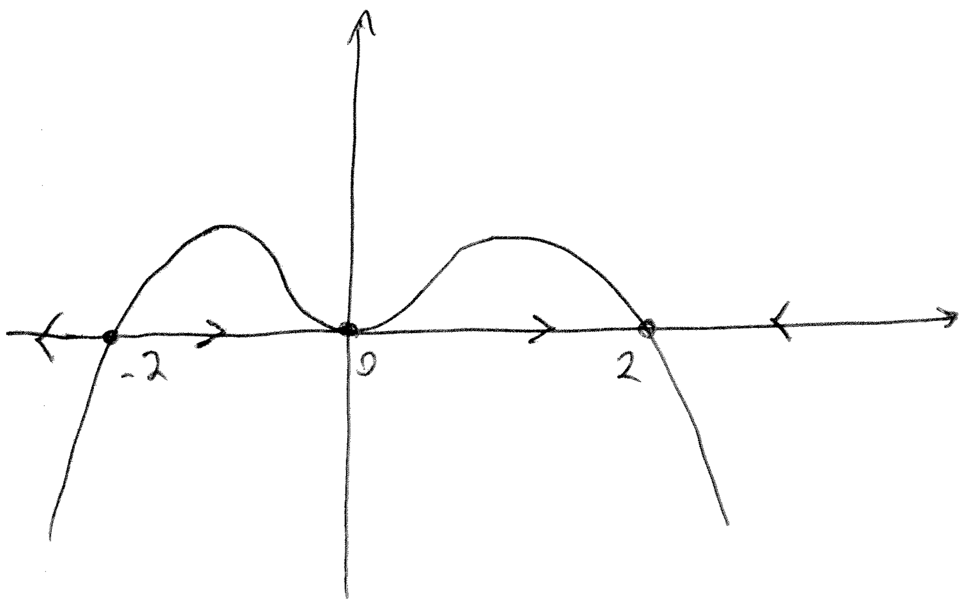
$N = -1$ stable

$N = 0$ unstable

$N = 1$ stable



14) $\frac{dN}{dt} = N^2(4-N^2) = N^2(2-N)(2+N)$



Equilibria

$N=-2$ unstable

$N=0$ semistable

$N=2$ unstable

15) $\frac{dN}{dt} = rN(1 - N/k) - h$

(a) Equilibria

$$rN\left(1-\frac{N}{k}\right) - h = 0$$

$$rN - \frac{r}{k}N^2 - h = 0$$

$$N^2 - kN + \frac{kh}{r} = 0$$

$$\Rightarrow N_1, N_2 = \frac{k}{2} \pm \sqrt{\left(\frac{k}{2}\right)^2 - \frac{kh}{r}}$$

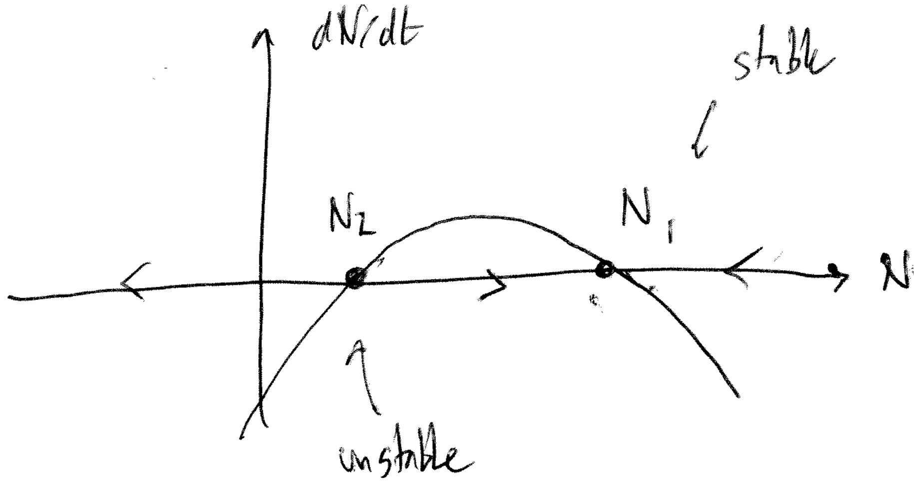
$$\left(\frac{k}{2}\right)^2 - \frac{kh}{r} > 0 \quad \text{in square root}$$

when $\left(\frac{k}{2}\right)^2 > \frac{kh}{r}$

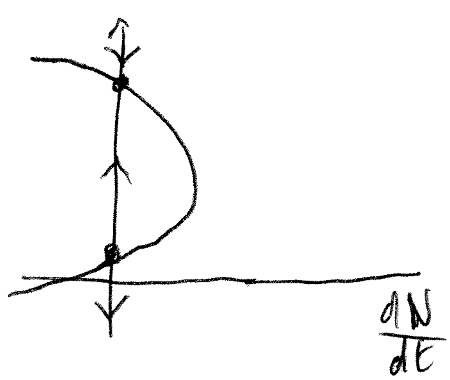
i.e. when $h < \frac{rk}{4}$

$\Rightarrow$ Two real equilibria

(b)



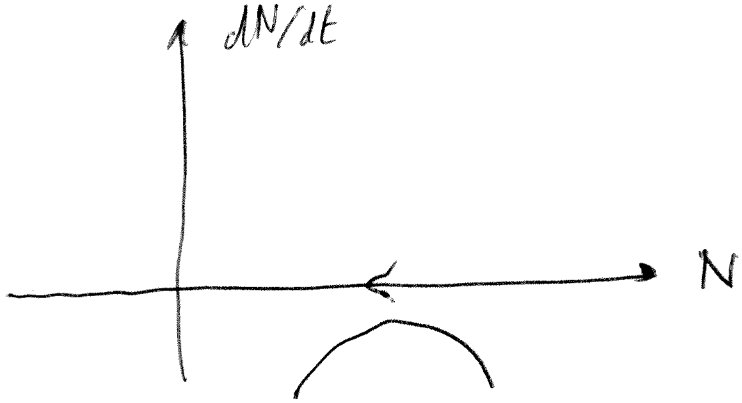
(c)

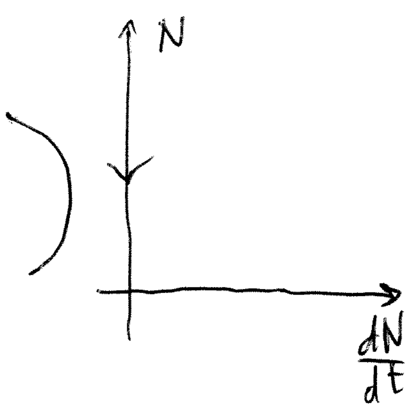


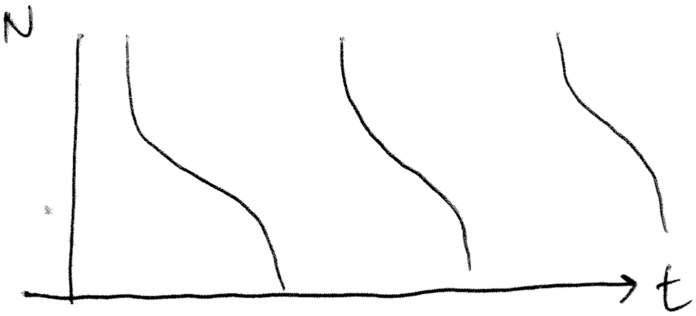
(d) If $h > \frac{rk}{4}$, then

$$\left(\frac{k}{2}\right)^2 - \frac{kh}{r} < 0$$

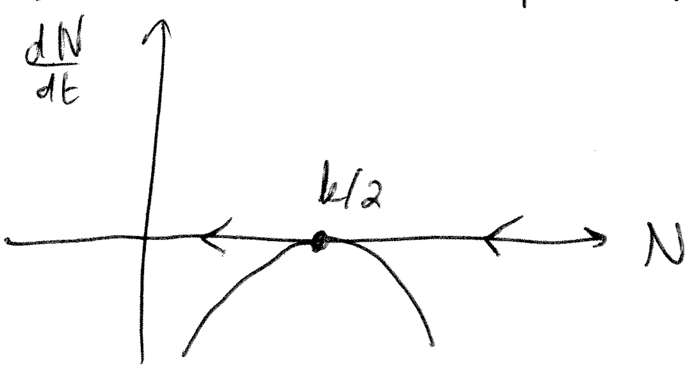
$\Rightarrow$ N_1 and N_2 are complex

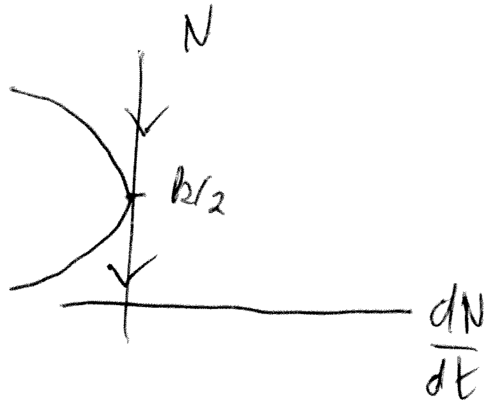


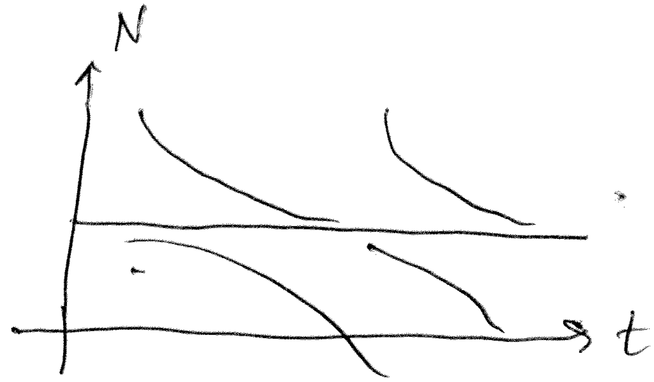




(e) $h = rk/4 \Rightarrow N_1 = N_2 = k/2$




N
k/2
dN
dt


N
t