MATH-2400 Sections 17–18 NAME: ____________________________

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Sample Exam #2

Please show all work. Each question is worth 10 points.
Do not use text books, notes, calculators, or other aids.
You may use one $8\frac{1}{2} \times 11$" crib sheet.

PROBLEM #	POINTS
1	
2	
3	
4	
5	
Total	

1. Using variation of parameters, find the general solution of the equation

$$x^2y'' - 2xy' + 2y = x^2, \qquad x > 0$$

Hom. Prob: Ind. Eq'n $r(r-1)-2r+2 = r^2-3r+2 = (r-2)(r-1) = 0 \Rightarrow r = 1, 2$

Comp. sol'n $y_c = C_1 x + C_2 x^2$.

V.O.P. assumption $y = u_1(x)\cdot x + u_2(x)\cdot x^2$

Then $u_1'(x)\cdot 1 + u_2'(x)\cdot 2x = \frac{x^2}{x^2} = 1$ (1)

$u_1'(x)\cdot x + u_2'(x)x^2 = 0$ (2)

$x\cdot(1) - (2) \Rightarrow u_2'(x)x^2 = x$, so $u_2'(x) = \frac{1}{x}$

$\Rightarrow u_2(x) = \log(x) + C_2$

From (2) $\frac{1}{x}\cdot x^2 + u_1'(x)x = 0 \Rightarrow u_1'(x) = -1$ so $u_1(x) = -x + C_1$

Thus $y(x) = u_1(x)\cdot x + u_2(x)\cdot x^2$

$= (-x + C_1)x + (\log(x) + C_2)x^2$

$= C_1 x - x^2 + C_2 x^2 + x^2\log(x)$

Relabel C_2 $\quad = C_1 x + C_2 x^2 + x^2 \log(x)$.

2. Some mass-spring system is described by the equation

$$\ddot{y} + ky = \cos t.$$

(a) Find the solution of this problem for $k < 0$ and $k = 0$.

(b) When $k > 0$, for which value(s) of k should we expect $y(t)$ to grow indefinitely in time as $t \to \infty$? You can either give a physical explanation, or explain what the solution looks like for special values of k.

(a) When $k=0$, $\ddot{y} = \cos(t)$, integrate twice to obtain $y(t) = -\cos(t) + C_1 t + C_2$.

If $k<0$, Hom. prob Char eqn is $r^2 + k = 0 \Rightarrow r = \pm\sqrt{-k}$ (Real, since $k<0$).

Then comp sol. is $y_c = C_1 e^{\sqrt{-k}t} + C_2 e^{-\sqrt{-k}t}$.

For part. guess $y_p = A\sin(t) + B\cos(t)$. $y_p'' = -A\sin(t) - B\cos(t)$, $ky = kA\sin(t) + kB\cos(t)$ $\Rightarrow$ $(k-1)A = 0$, $(k-1)B = 1$ $\Rightarrow$ $A = 0$, $B = \frac{1}{k-1}$ (ok since $k<0$)

So gen. sol'n is $y(t) = \frac{1}{k-1}\cos(t) + C_1 e^{\sqrt{-k}t} + C_2 e^{-\sqrt{-k}t}$.

(b) ~~Resonant~~ Natural freq. occurs at $\omega_0 = \sqrt{\frac{k}{m}}$. Resonant if $\omega = 1 = \omega_0 = \sqrt{\frac{k}{m}}$. Since $m=1$, $\boxed{k=1.}$

This can also be solved from method of undet. coeffs as well (Char. eqn is still $r^2 + k = 0$). For $k=1$, $\alpha + i\beta = i$ solves char eqn, meaning that part sol'n takes form $\underline{\underline{t}}(A\cos(t) + B\sin(t))$. This factor of t is unbounded as $t \to \infty$)

3. Find the general solution of the linear system

$$\dot{\mathbf{x}} = \underbrace{\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}}_{=A} \mathbf{x},$$

and sketch the phase portrait in the $x_1 - x_2$ plane. Along which vector(s) do solutions tend towards the origin as $t \to \infty$?

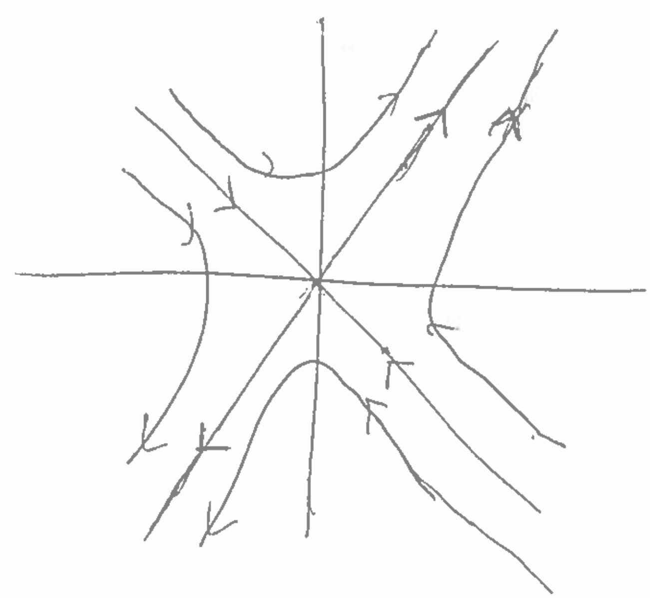
$\begin{pmatrix} 1-r & 2 \\ 3 & 2-r \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad \det(A-Ir) = (1-r)(2-r)-6 = r^2-3r-4 = (r-4)(r+1) = 0.$

$\Rightarrow r = 4, -1$

$r=4$ $\begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow -3v_1 + 2v_2 = 0$, $v_2 = \frac{3v_1}{2}$ $\quad z^{(1)} = \begin{pmatrix} 1 \\ \frac{3}{2} \end{pmatrix}$

$r=-1$ $\begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 2v_1 + 2v_2 = 0 \Rightarrow v_1 = -v_2$, $z^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Sol: $x(t) = c_1 e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{4t}\begin{pmatrix} 1 \\ \frac{3}{2} \end{pmatrix}$



Vectors travel to origin along $z^{(2)}$

4. Compute the general solution of the system

$$\dot{\mathbf{x}} = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} \mathbf{x},$$

and sketch the phase portrait in the $x_1 - x_2$ plane. What happens to the solutions as $t \to \infty$?

$\begin{pmatrix} -1-r & 1 \\ -1 & -1-r \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$, $\det(A - Ir) = (-1-r)^2 + 1 = (1+r)^2 + 1 = 0 \Rightarrow (1+r) = \pm i$, or $r = -1 \pm i$

Use $r = -1 + i \cdot 1$, $\lambda = -1$, $\mu = 1$

$\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow -iv_1 + v_2 = 0 \Rightarrow v_2 = iv_1$, so $z = \begin{pmatrix} 1 \\ i \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{a} + i\underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{b}$

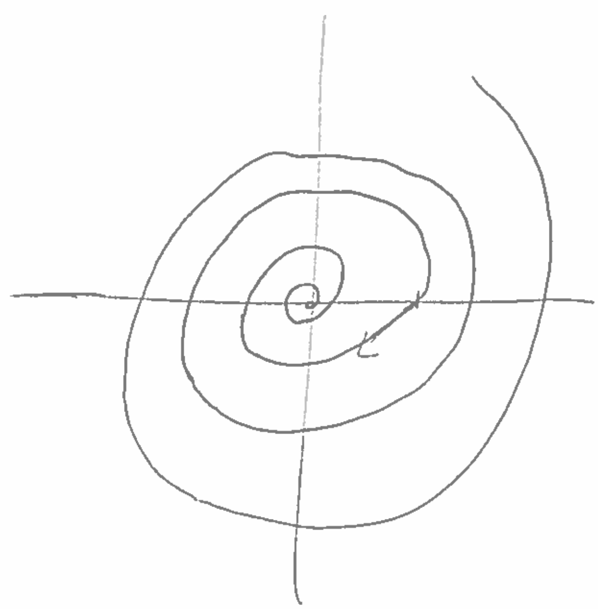
Then sol'n is

$$x(t) = c_1 e^{-t}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos(t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin(t)\right) + c_2 e^{-t}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\sin(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos(t)\right)$$

Inward spiral (stable), since $\lambda < 0$.

Vel. at $(1,0)$ is

$$\dot{x} = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$



5. For any real $a_1 > a_2 > 0$, compute the eigenvalues and eigenvectors of the *diagonal* matrix

$$A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$$

Bonus: What about the *antidiagonal* matrix

$$B = \begin{pmatrix} 0 & a_1 \\ a_2 & 0 \end{pmatrix}.$$

$$\det(A - Ir) = (a_1 - r)(a_2 - r) = 0 \Rightarrow r = a_1, a_2 \leftarrow \text{E-vals}$$

$$r = a_1 \quad \begin{pmatrix} 0 & 0 \\ 0 & a_2 - a_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow (a_2 - a_1) v_2 = 0, \text{ or } v_2 = 0, \text{ Then } \vec{z}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$r = a_2 \quad \begin{pmatrix} a_1 - a_2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow (a_1 - a_2) v_1 = 0, \text{ or } v_1 = 0, \text{ Then } \vec{z}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Bonus $\quad \det(B - Ir) = \begin{vmatrix} -r & a_1 \\ a_2 & -r \end{vmatrix} = r^2 - a_1 a_2 = 0 \Rightarrow r = \pm\sqrt{a_1 a_2}$

$$r = \sqrt{a_1 a_2} \quad \begin{pmatrix} \sqrt{a_1 a_2} & a_1 \\ a_2 & \sqrt{a_1 a_2} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \vec{0}, \text{ so } \sqrt{a_1 a_2}\, v_1 + a_1 v_2 = 0 \Rightarrow \sqrt{a_2}\, v_1 + \sqrt{a_1}\, v_2 = 0 \Rightarrow \vec{z}^{(1)} = \begin{pmatrix} 1 \\ -\sqrt{\frac{a_2}{a_1}} \end{pmatrix}$$

$$r = -\sqrt{a_1 a_2} \quad \begin{pmatrix} -\sqrt{a_1 a_2} & a_1 \\ a_2 & -\sqrt{a_1 a_2} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \vec{0}, \text{ so } -\sqrt{a_1 a_2}\, v_1 + a_1 v_2 = 0 \Rightarrow \sqrt{a_1}\, v_2 = +\sqrt{a_2}\, v_1 \Rightarrow \vec{z}^{(2)} = \begin{pmatrix} 1 \\ \sqrt{\frac{a_2}{a_1}} \end{pmatrix}$$