MATH-2400 (Sections 17-18, Spring 2017) NAME: ______________________

Instructor: Joe Klobusicky SECTION: ______________________

Practice Exam

- Please show all work.
- Do not use text books, notes, calculators, cell phones, or other aids.
- You may use one side of an $8\frac{1}{2} \times 11$" crib sheet.
- Each problem is 10 out of a total of 60 points.

PROBLEM #	POINTS
1	
2	
3	
4	
5	
6	
Total	

1. Solve the initial-value problem

$$(x+1)^2 y' - (x+1)y = -2, \qquad y(0) = 0,$$

for y as a function of ~~t~~ x and sketch the solution.

$$y' - \frac{y}{x+1} = -\frac{2}{(x+1)^2}$$

$$\mu(x) = e^{-\int \frac{1}{x+1}dx} = e^{-\log|x+1|} = \frac{1}{x+1}$$

$$(\mu(x)y)' = \mu(x)\left(-\frac{2}{(x+1)^2}\right)$$

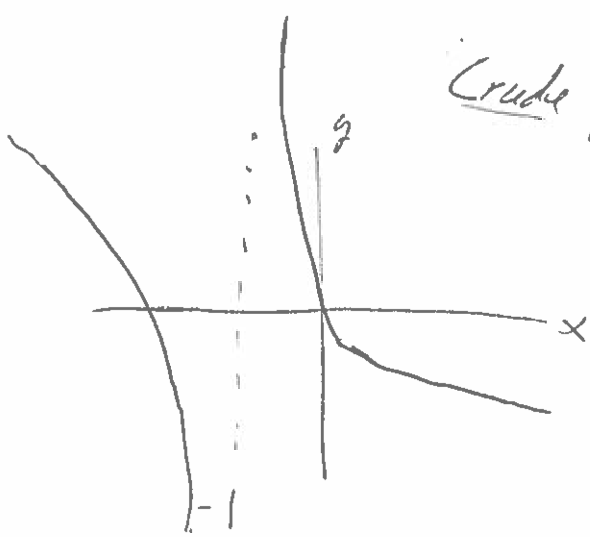
$$\Rightarrow \frac{y}{x+1} = -\int \frac{2\,dx}{(x+1)^3} + C = \frac{1}{(x+1)^2} + C$$

$$y = \frac{1}{x+1} + C(x+1)$$

From ICs

$$0 = \frac{1}{0+1} + C(0+1) \Rightarrow C = -1$$

So $y = \frac{1}{x+1} - x+1$



Crude graph of $y(x)$

2. Suppose that 100 mg of thorium-234 are initially present in a closed container, and that thorium-234 is added to the container at a constant rate of 1 mg/day.

(a) Find the amount $Q(t)$ of thorium-234 in the container at any time, given that its decay rate is 0.01 days^{-1} (Hint: the change in $Q(t)$ is the difference between rate added and rate decayed). .

(b) Find the limiting amount Q_1 of thorium-234 in the container as $t \to \infty$.

(c) If thorium-234 is added to the container at a rate of k mg/day, find the value of k that is required to maintain a constant level of 100 mg of thorium-234.

(a) IC $Q(0)=100$ $\quad Q'(t) = \underbrace{1}_{\text{Rate in}} - \underbrace{.01Q}_{\text{Rate out}}$

Solve $Q'+.01Q=1$ $\quad \mu(t)=e^{.01t}$, so $(e^{.01t}Q)'=e^{.01t}$

$\Rightarrow e^{.01t}Q=\frac{1}{.01}e^{.01t}+C=100e^{.01t}+C$, meaning $Q(t)=100+Ce^{-.01t}$

By IC $100=100+C \Rightarrow C=0$ (!!) so $Q(t)=100$ is constant solution!

(b) Well, since it is constant, limiting sol'n is also 100. This holds for any other initial value as well (verify this).

c) Here $Q'(t)=K-.01Q$. Eq. if $Q'=0=K-.01Q(0)$

$\Rightarrow K=.01(100)=1$, which was the rate we had before.

For more practice, use init. cond. of $Q(0)=200$ and $Q(0)=50$.

3. Find the general form for the solution y of the initial-value problem

$$y'' + y' + 2y = \underbrace{\sin(3x)e^{x}(1+x)}_{g_1} + \underbrace{x}_{g_2}.$$

Note: By general form, this means you may leave unsolved constants in your solution.

Hom. Prob.

Char Eqn $r^2 + r + 2 = 0 \Rightarrow r_{1,2} = \frac{-1 \pm \sqrt{1-8}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{7}}{2}$

So $y_c = c_1 e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{7}}{2}x\right) + c_2 e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{7}}{2}x\right)$

g_1 $\sin(3x)e^{x}(1+x)$

$P^n e^{\alpha x} \sin(\beta x)$ $n=1$ $\alpha=1$ $\beta=3$, $1+3i$ not sol'n of ch. eqn $(s=0)$

So $y_{p_1} = (Ax+B)e^{x}\sin(3x) + (Cx+D)e^{x}\cos(3x)$

g_2 P_n, $n=1$ 0 not soln of ch. eqn $(s=0)$

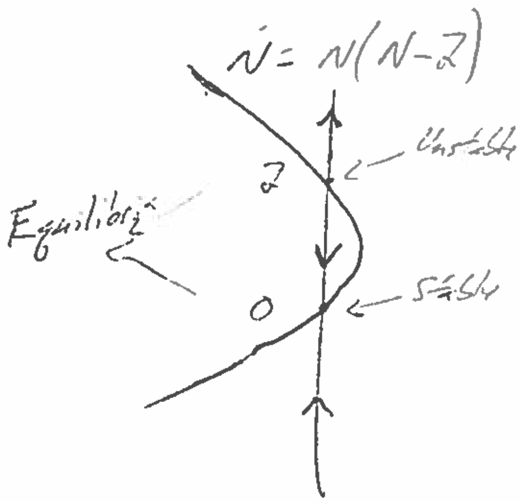
$y_{p_2} = (Ex+F)$

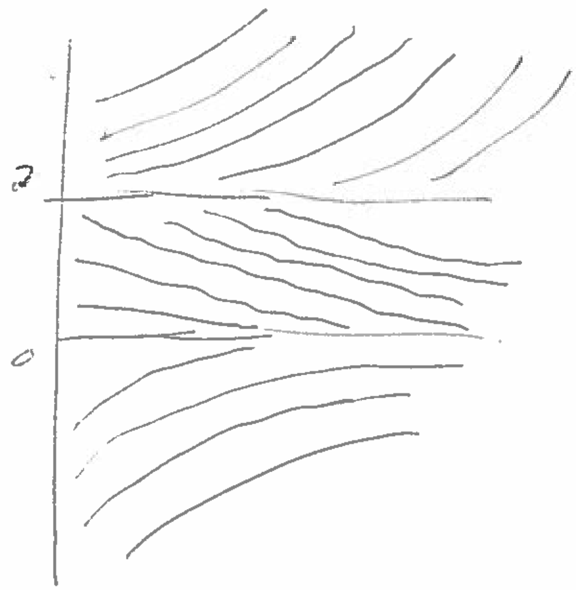
Then $y = y_c + y_{p_1} + y_{p_2}$.

4. Consider the problem

$$\frac{dN}{dt} = N^2 - 2N.$$

Sketch dN/dt versus N, determine all possible fixed equilibria and their stability type (stable, unstable, semi-stable), and draw some representative integral (solution) curves in the $t - N$ plane.





5. Compute the solution of the initial-value problem

$$x^2y'' + xy' + 4y = 0, \quad x > 0.$$

As $x \to \infty$, will the solution $y(x)$ approach ∞, $-\infty$, or neither? Explain why? (Note: this answer should not depend on initial conditions!)

Eul. Eqn, so ind eqn is $r(r-1)+r+4=0$

$r^2+4=0$ so $r_{1,2}=\pm 2i$

So $y=|x|^{0}\left[c_1\cos(2\ln(x))+c_2\sin(2\ln(x))\right]$

Since $|\sin(x)|<1$, $|\cos(x)|<1$, sol'n are bound between

$-(|c_1|+|c_2|)$ and $(|c_1|+|c_2|)$

6. Compute the solution of the initial-value problem

$$y'' + 2y' + 5y = 0, \qquad y(0) = 1, \quad y'(0) = -1.$$

and sketch this solution.

Ch. eqn $r^2 + 2r + 5 = 0$

$$\frac{-2 \pm \sqrt{4 - 5(4)}}{2} = -1 \pm 2i$$

so $y(x) = C_1 e^{-x} \sin(2x) + C_2 e^{-x} \cos(2x)$

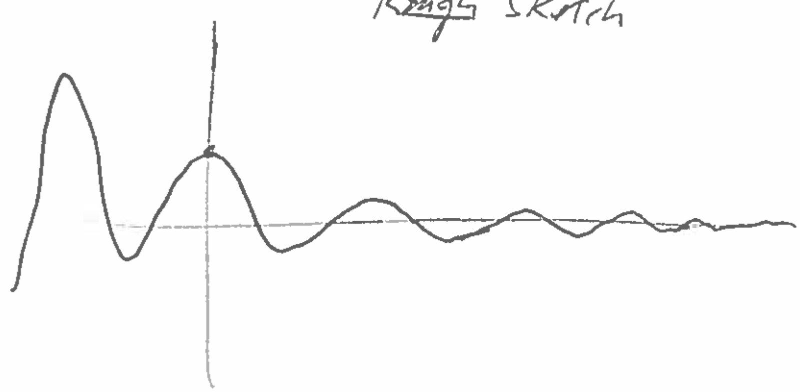
From $y(0) = 1$, $1 = C_2$

$y'(x) = C_1\left(e^{-x}(2\cos(2x)) + (-e^{-x}\sin(2x))\right)$
$\quad + C_2\left(e^{-x}(-2\sin(2x)) + (-e^{-x}\cos(2x))\right)$

so $-1 = C_1(2) + C_2(-1) = 2C_1 - 1 \Rightarrow C_1 = 0$

so $y(x) = e^{-x}\cos(2x)$

Rough Sketch



(Extra page for work)