MATH 2400
HOMEWORK 1 SOLUTIONS

1) $y' = x + y$

Highest order derivative : 1

Number of indep. variables : 1 , x

$\Rightarrow$ First order ODE

2) $xy'' + y = \sin x$

Highest order derivative : 2

Number of indep. variables : 1 , x

$\Rightarrow$ Second order ODE

3) $u_x = u_{yy}$

Highest order deriv. : 2

Number of indep. vars : 2 , x and y

$\Rightarrow$ Second order PDE

4) $\dfrac{\partial^3 y}{\partial x^3} + \dfrac{\partial^2 y}{\partial z^2} = y$

Highest order deriv : 3

Number of indep. vars : 2 , x and z

$\Rightarrow$ third order PDE

5) $y' + y = xe^{-x} + 1$

$p(x) = 1$ $\quad$ $g(x) = xe^{-x} + 1$

$\Rightarrow \mu(x) = e^{\int p(x)dx} = e^{\int dx} = e^x$

$\Rightarrow \mu(x)(y'+y) = [\mu(x)y]' = [e^x y]' = \mu(x)g(x)$

$= x + e^x$

$\Rightarrow$ Integrate both sides

$$e^x y = \int x + e^x \, dx$$

$$= \frac{x^2}{2} + e^x + C$$

$$\Rightarrow \quad y = \frac{x^2}{2}e^{-x} + 1 + Ce^{-x}$$

6) $xy' + 2y = \sin x$

OR IN STANDARD FORM $\quad y' + \frac{2}{x}y = \frac{\sin x}{x}$

$p(x) = \frac{2}{x}$ $\quad$ $g(x) = \frac{\sin x}{x}$

$\Rightarrow \mu(x) = e^{\int \frac{2}{x}dx} = e^{2\log x} = e^{\log x^2} = x^2$

$\Rightarrow [x^2 y]' = x^2\left(\frac{\sin x}{x}\right) = x \sin x$

$\Rightarrow \quad x^2 y = \int x \sin x \, dx$

$$= -x\cos x + \int \cos x\,dx \quad \text{(Integration by parts)}$$

$$= -x\cos x + \sin x + C$$

$$\Rightarrow y = -\frac{\cos x}{x} + \frac{\sin x}{x^2} + \frac{C}{x^2}$$

7) $y' + \underset{\uparrow\, p(x) = 2x}{2x}y = \underset{\uparrow\, g(x) = 2xe^{-x^2}}{2xe^{-x^2}}$

$$\Rightarrow \mu(x) = e^{\int 2x\,dx} = e^{x^2}$$

$$\Rightarrow [e^{x^2}y]' = e^{x^2}(2xe^{-x^2})$$

$$= 2x$$

$$\Rightarrow e^{x^2}y = \int 2x\,dx = x^2 + C$$

$$\Rightarrow y = (x^2 + C)e^{-x^2}$$

8) $(1+x^2)y' + 4xy = \frac{1}{(x^2+1)^2}$

OR $y' + \underset{\uparrow\, p(x)}{\frac{4x}{(1+x^2)}}y = \underset{g(x)}{\frac{1}{(1+x^2)^3}}$

$$\Rightarrow \mu(x) = e^{\int p(x)\,dx} = e^{\int \frac{4x}{1+x^2}\,dx}$$

$$= e^{2\log(1+x^2)} = e^{\log(1+x^2)^2}$$

$$= (1+x^2)^2$$

$$\Rightarrow [(1+x^2)^2 y]' = (1+x^2)^2 \frac{1}{(1+x^2)^3} = \frac{1}{1+x^2}$$

$$\Rightarrow (1+x^2)^2 y = \arctan x + C$$

$$\Rightarrow y = \frac{\arctan x + C}{(1+x^2)^2}$$

9) $y' + \frac{2}{x} y = \frac{\cos x}{x^2}$ $\qquad y(\pi) = 0$

$$\mu(x) = e^{\int \frac{2}{x}\,dx} = e^{2\log x} = e^{\log x^2} = x^2$$

$$\Rightarrow [x^2 y]' = x^2 \left(\frac{\cos x}{x^2}\right) = \cos x$$

$$\Rightarrow x^2 y = \int \cos x\,dx = \sin x + C$$

$$\Rightarrow y = \frac{\sin x + C}{x^2}$$

$\longrightarrow$

$$y(\pi) = \frac{\sin\pi + C}{\pi^2} = \frac{C}{\pi^2} = 0 \quad \Rightarrow C = 0$$

$$\Rightarrow \quad y = \frac{\sin x}{x^2}$$

10) $x^2 y' + 3xy = \frac{\sin x}{x}$ $\qquad y(\pi) = 0$

OR $\quad y' + \frac{3}{x} y = \frac{\sin x}{x^3}$

$$\Rightarrow \mu(x) = e^{\int \frac{3}{x} dx} = e^{3\log x} = x^3$$

$$\Rightarrow [x^3 y]' = \sin x$$

$$\Rightarrow \quad x^3 y = \int \sin x \, dx = -\cos x + C$$

$$\Rightarrow \quad y = \frac{C - \cos x}{x^3}$$

$$y(\pi) = \frac{C - \cos\pi}{3} = \frac{C+1}{\pi^3} = 0 \quad \Rightarrow C = -1$$

$$\Rightarrow \quad y = \frac{-1 - \cos x}{x^3}$$

11) $xy' + y = e^x \qquad y(1) = 1$

OR $\quad y' + \frac{1}{x} y = \frac{1}{x} e^x$

$\Rightarrow \quad \mu(x) = e^{\int \frac{1}{x} dx} = e^{\log x} = x$

$\Rightarrow \quad [xy]' = e^x$

$\Rightarrow \quad xy = \int e^x dx = e^x + C$

$\Rightarrow \quad y = \dfrac{e^x + C}{x}$

$y(1) = \dfrac{e + C}{1} = 1 \Rightarrow C = 1 - e$

$\Rightarrow y = \dfrac{e^x + 1 - e}{x}$

12) $y' + y = \dfrac{1}{1+x^2} \qquad y(0) = 0$

$\mu(x) = e^{\int dx} = e^x$

$\Rightarrow \quad [e^x y]' = \dfrac{e^x}{1+x^2}$

t is dummy variable

$\Rightarrow \quad e^x y = \displaystyle\int_{x_0}^{x} \frac{e^t}{1+t^2} dt + C$

by Fund. thm of calculus

(no explicit expression
for integral

$$\Rightarrow y = e^{-x}\int_{x_0}^{x} \frac{e^t}{1+t^2}\,dt + e^{-x}C$$

$$y(0) = e^0\int_{x_0}^{0} \frac{e^t}{1+t^2}\,dt + e^0 C = 0$$

$$\Rightarrow C = \int_0^{x_0} \frac{e^t}{1+t^2}\,dt$$

$$\Rightarrow y = e^x\left(\int_{x_0}^{x} \frac{e^t}{1+t^2}\,dt + \int_0^{x_0} \frac{e^t}{1+t^2}\,dt\right)$$

$$= e^x \int_0^x \frac{e^t}{1+t^2}\,dt$$

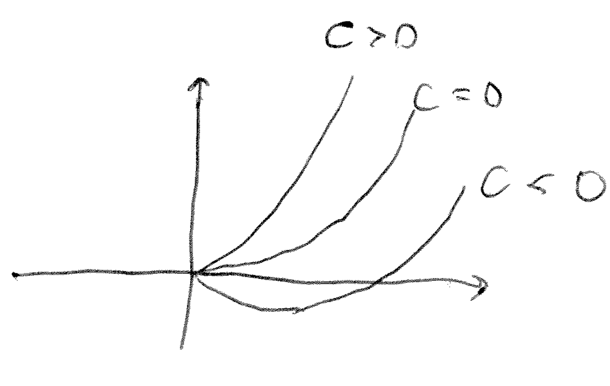
13) $y' - \frac{1}{x}y = x$

$$u(x) = e^{\int -\frac{1}{x}dx} = e^{-\log x} = e^{\log 1/x} = 1/x$$

$$\Rightarrow \left[\frac{y}{x}\right]' = \frac{x}{x} = 1$$

$$\Rightarrow \frac{y}{x} = \int dx = x + C$$

$$\Rightarrow y = x^2 + Cx$$



As $x \to 0$, $y \to 0$ for
all C