Kinetic Limits of Piecewise Deterministic Markov Processes

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1 Introduction

1.1 Two dimensional grain boundary networks

Our purpose in this series of articles is to propose a new class of stochastic models that describe the statistics of an evolving cellular network. While the main contributions of this article are not restricted to cellular networks, we find it valuable to begin with an example that is central to the story.

A two dimensional grain boundary network consists of smooth arcs such that: (i) the normal velocity of each arc is proportional to its curvature (*curvature flow*); (ii) edges (typically) meet at trivalent junctions at an angle of $2\pi/3$ (the *Herring boundary condition*) [8]. Such a network of arcs decomposes the plane into a disjoint collection of grains that have the topology of polygons. Condition (ii) expresses the equilibrium of line tensions at a junction. Its precise formulation requires some care since more than three edges may meet at a junction when grains or grain boundaries shrink to a point. The time evolution of such a network will be called *grain boundary evolution*.

An important aspect of grain boundary evolution is the celebrated von Neumann-Mullins relation: the area $a(t)$ of a grain with s-sides (a topological s-gon) changes linearly in time

$$\frac{da}{dt} = c(s-6), \tag{1}$$

where c is a material constant depending on surface tension and grain mobility. Thus, the geometry of each grain does not affect its growth, all that matters is the topology. In this setting, the statistics of a network with many grains are naturally described by a set of *number densities* $f_s(t,a)$ that count the number of s-gonal cells per unit area that have area a at time t. We derive kinetic equations that describe the evolution of $f_s(t,a)$ from a simpler stochastic particle system that includes a deterministic drift (as in equation (1)) along with stochastic 'switching' rules between populations based on the geometry of grain boundary networks. Our model is an instance of a Piecewise Deterministic Markov Process (PDMP). The contributions in our work include the development of such models from the geometry of grain boundary networks, a rigorous hydrodynamic limit

theorem that establishes a kinetic limit of PDMPs, and the rigorous analysis of the asymptotic behavior of one such kinetic limit.

1.2 Universality in cellular networks

The modern study of cellular networks in materials science originates in the pioneering work of C.S. Smith (e.g. [16]). His early experiments on a froth of soap bubbles have been revisited several times [7, 17]. In the past ten years, early numerical experiments on froths [18, 11, 13] have been supplanted by large scale computations of grain boundary networks [1, 4, 12]. An astonishing feature of these computations is the dynamic emergence of 'universal statistics' of networks. In order to understand this idea, note that there are many different ways of constructing random cellular networks in the plane, both in theory and numerical experiment. For instance, a random network may be chosen by first generating a point process, and then constructing the Voronoi diagram for this process. Each such random network may be evolved under curvature flow with the Herring boundary condition. There is no a priori reason, to expect that the number densities $f_s(t, a)$ from statistically distinct initial networks (e.g. from different point processes) will asymptote to limiting distributions as $t \to \infty$ after a suitable rescaling, but surprisingly this appears to be the case.

Rigorous proof of the emergence of 'universal cellular networks' under grain boundary (or similar geometric evolution) is currently out of reach, since we lack both a detailed understanding of random trivalent networks embedded in the plane, as well as a well-posedness theory for grain boundary evolution that allows flow through 'singular' events such as the vanishing of grains. Mean-field kinetic models seem more amenable to analysis . Our work is motivated in part by considerable recent progress in the analysis of universal scaling behavior for the related problem of Ostwald ripening. Mathematical results in this area include both proofs of convergence to self-similar form, as well as derivation of basic mean-field kinetic equations, such as the LSW model, from particle systems (see [14] for a review). In contrast, the few kinetic models that exist for grain boundary networks have been derived in a somewhat *ad hoc* manner [6, 5, 11, 13], and the only rigorous analysis of dynamic scaling is very recent [9].

1.3 Outline

In this series of articles, we propose and analyze a class of stochastic particle systems that bridge the gap between numerical simulation and kinetic theory of evolving cellular networks. The work consists of three parts:

1. Kinetic limits of piecewise deterministic Markov processes.
2. Derivation of new kinetic equations from topological rules; comparison with numerical simulations.
3. Analysis of the long-time behavior of the kinetic equations in an example.

The articles have been written so that they may be read independently for the most part (though they are related in theme and content as discussed above). In technical terms, the first article is most closely tied to fluid-limits in queuing theory and the theory of piecewise-deterministic Markov processes. It can be viewed as a demonstration of the utility of these methods for cellular networks. In the second article, we use a lemma on the topology of evolving networks to provide a consistent derivation of new kinetic equations for two-dimensional grain boundary networks. The main convergence theorem of Paper 1 then yields kinetic equations as a rigorous hydrodynamic limit. We also provide a numerical comparison between our model and recent numerical simulations of grain boundary networks. In the final paper, we develop a well-posedness theory for the limit equation in a simple instance of our particle model (a *dynamic shuffler*) and explore the long-time asymptotics. This paper demonstrates some interesting analytical challenges implicit in our approach.

2 Statement of results

2.1 The model

We consider a system of $N(t)$ particles at time t distributed amongst M species. Each particle is of the form (s, x) where $s = 1, \ldots, M$, indexes the species and $x \in \mathbb{R}_+$ denotes the size of the particle. The total number of particles in each species is denoted $N_s(t)$, thus $N(t) = \sum_{s=1}^{M} N_s(t)$. The letter N (without the argument t) is always used to mean $N(0)$, and is a measure of the size of the system. The state of the system is denoted

$$(\mathbf{s}, \mathbf{x}) = (s_1, \ldots, s_{N(t)}; x_1, \ldots, x_{N(t)}). \tag{2}$$

The evolution of the system consists of a deterministic flow interspersed with stochastic jumps. We describe these in turn.

The deterministic flow is motivated by the von Neumann-Mullins rule[1]. We divide the species into three distinct groups S_-, S_0 and S_+ of size M_-,M_0 and M_+ respectively, with $M = M_- + M_0 + M_+$. It is convenient to label these species in order:

$$S_- = \{1, \ldots, M_-\}, S_0 = \{M_- + 1, \ldots, M_- + M_0\}, S_+ = \{M_- + M_0 + 1, \ldots, M\}. \tag{3}$$

For each species $s \in S_-$, we assume given a constant velocity $v_s < 0$, so that a particle of size x at $t = 0$ has size $x - v_s t$ at time $t > 0$. The first *exit time* is the time at which the size of the particle vanishes, namely

$$T_s(x) = \frac{x}{v_s}, \quad s \in S_-. \tag{4}$$

[1]It is possible to generalize the results presented here to the case when the drift is not constant. In that case, we must assume that that for each species s, there is a globally Lipschitz velocity field $v_s : \mathbb{R}_+ \to \mathbb{R}$ in order that the main regularity estimate that provides compactness continues to hold. We avoid such generality since it obscures the main ideas presented here, and since constant velocity fields suffice for 2D grain boundary coarsening

We assume that the species $s \in S_0$ do not drift. That is, $v_s = 0$ for $s \in S_0$. Finally, we assume $v_s > 0$ for $s \in S_+$. The first exit time for all particles of species S_0 and S_+ is $+\infty$.

Randomness is introduced into the system in the following way. As t increases, each particle (s, x) in the system drifts deterministically $(s, x) \mapsto (s, \varphi_s(x, t))$ until one of the following *critical events* occur:

(B) *Boundary event:* A particle hits the origin, i.e. $\varphi_s(x, t) = 0$ for some (s, x) with $s \in S_-$ and $t = T_s(x)$.

(I) *Interior event:* An independent Poisson-β clock attached to each particle rings. Here $\beta > 0$ is a fixed parameter.

To fix ideas, we illutrate these definitions in the context of grain boundary networks. Here the state of the system is a collection of $N(t)$ grains, each belonging to one of M topological classes; s denotes the number of sides of a grain and x denotes its area. The velocity field v_s governing the evolution of an s-gon is given by the von Neumann-Mullins rule (1). The critical events correspond respectively to: (B) the removal of a grain from the network when its size shrinks to zero; and (I) a random interchange of grains of different topology when an edge vanishes.

Though the size of each particle evolves deterministically, each boundary and interior event gives rise to a random mutation of particles of different species. We model each mutation with a *mutation matrix*. There are $M_- + 1$ such matrices: M_- matrices corresponding to the M_- possible boundary events at species $l \in S_-$, and one matrix for interior events. We find it necessary to include mutations in such generality to account for the topology of cellular networks – the topological changes arising from the vanishing of 3, 4 and 5 sides grains in grain boundary networks is *not* the same. Aside from some notational complexity, such generality does not affect the analysis. In paper 2, we explain how to choose the mutation rules based on the geometry of planar grain boundary networks.

Consider the boundary event when a single particle of species l hits the origin. The corresponding mutation is determined by a positive integer $K^{(l)}$, an $M \times K^{(l)}$ mutation matrix $R^{(l)}$ taking values in $1, \dots, M$, and a fixed M-vector $w^{(l)}$ with positive entries. We choose $K^{(l)}$ particles and mutate them as follows. First, $K^{(l)}$ iid integers $S_1, \dots, S_{K^{(l)}}$ that index species are chosen with probability proportional to the weights $w^{(l)}$ and the total population of each species:

$$\mathbb{P}(S = k) = \frac{w_k^{(l)} N_k(t)}{\sum_{p=1}^{M} w_p^{(l)} N_p(t)}, \quad k = 1, \dots, M. \tag{5}$$

Second, for each random species S_j, a random size X_j is chosen with equal probability $1/N_{S_j}(t)$ amongst the sizes of all the particles of species S_j. Finally, these random particles are mutated as follows:

$$(S_j, X_j) \longmapsto \left(R^{(l)}_{S_j, j}, X_j\right), \quad j = 1, \dots, K^{(l)}. \tag{6}$$

Thus, a particle of species S_j with size X_j is lost, and a particle of species $R^{(l)}_{S_j,j}$ with the size X_j is created in the mutation. See Fig. 1 for an example with four species. In the degenerate event that the sizes of p species, $p > 1$, hit the origin simultaneously, we repeat the process above p-times, ordering the boundary events at species $l_1, \ldots, l_p$ in the sequence $l_1 \leq l_2 \leq \ldots \leq l_p$ to be definite. Such 'collisions' occur with zero probability in the kinetic limit.

The process of mutation at an interior event is similar. No particle vanishes, but particles are mutated according to a fixed positive integer $K^{(0)}$, a mutation matrix $R^{(0)}$ and weight $w^{(0)}_s$ as above. The integers $S_1, \ldots, S_{K^{(0)}}$ are chosen with probability

$$\mathbb{P}\left(S = k\right) = \frac{w^{(0)}_k N_k(t)}{\sum_{p=1}^{M} w^{(0)}_p N_p(t)}, \quad k = 1, \ldots, M, \tag{7}$$

and the particles mutated as follows

$$(S_j, X_j) \longmapsto \left(R^{(0)}_{S_j,j}, X_j\right), \quad j = 1, \ldots, K^{(0)}. \tag{8}$$

In order to keep track of the flux in and out of a species we define the (constant) matrices with integer entries

$$J^{(l)}_{\sigma k} = \sum_{j=1}^{K^{(l)}} \mathbf{1}_{\{R^{(l)}_{kj} = \sigma\}}, \quad J^{(0)}_{\sigma k} = \sum_{j=1}^{K^{(0)}} \mathbf{1}_{\{R^{(0)}_{kj} = \sigma\}}. \tag{9}$$

Here k, and σ index species in $\{1, \ldots, M\}$, l indexes a species in S_-, and the entry $J^{(l)}_{k,\sigma}$ counts the total number of mutations from species k to species σ when a particle of species l hits the origin. Similarly, $J^{(0)}_{k,\sigma}$ enumerates the total number of mutations from species k to species σ at an interior event. We assume that there are no trivial mutations from a species to itself, i.e.,

$$J^{(l)}_{\sigma\sigma} = 0, \quad J^{(0)}_{\sigma\sigma} = 0. \tag{10}$$

Summing over all species we obtain the identities

$$\sum_{\sigma=1}^{M} J^{(l)}_{\sigma k} = \sum_{j=1}^{K^{(l)}} \sum_{\sigma=1}^{M} \mathbf{1}_{R^{(l)}_{kj} = \sigma} = \sum_{j=1}^{K^{(l)}} 1 = K^{(l)}, \quad \sum_{\sigma=1}^{M} J^{(0)}_{\sigma k} = K^{(0)}. \tag{11}$$

However, we do not assume detailed balance of mutations between species. That is, in general,

$$\sum_{k=1}^{M} J^{(l)}_{\sigma k} \neq K^{(l)}, \quad \sum_{k=1}^{M} J^{(0)}_{\sigma k} \neq K^{(0)}. \tag{12}$$

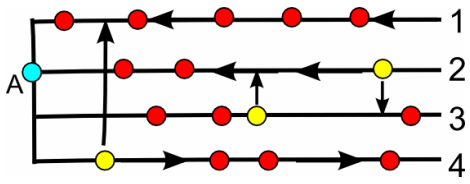

Figure 1: **A PDMP on four species**. Particles travel on four separate copies of $\mathbb{R}_+$. Velocity directions are represented by horizontal arrows, with species 3 having zero velocity. A boundary event occurs when a particle (labelled by "A") hits the origin. Here, three particles are then randomly selected ($K^{(2)} = 3$), and reassigned to different species by predetermined reassignments (given by vertical arrows). In this example, $R^{(2)}_{41} = 1, R^{(2)}_{32} = 2$, and $R^{(2)}_{23} = 3$.

2.2 Kinetic equations

Several important kinetic equations arise as hydrodynamic limits of Markovian particle models as the number of particles $N \to \infty$. The particle process described above is an example of a *piecewise deterministic Markov process* (PDMP). Roughly, the theory of PDMPs augments the structure of pure-jump Markov process to include a deterministic drift as above. The theory of PDMPs will imply that the particle system defined informally above generates a well-defined evolution. We defer a formal statement of this fact in order to state our main theorem.

For each state $(\mathbf{s}(t), \mathbf{x}(t))$ and a species $k \in \{1, \dots, M\}$ we define an empirical measure

$$\mu^N_k(t) = \frac{1}{N} \sum_{i=1}^{N(t)} \mathbf{1}_{\{s_i = k\}} \delta_{x_i}. \tag{13}$$

It is important to note that the empirical measures are normalized by the fixed initial number $N = N(0)$, not $N(t)$. Thus, $\sum_{k=1}^N \mu^N_k(t)$ is in general not a probability measure for $t > 0$.

Our hydrodynamic limit theorem (Theorem 1 below) shows that the empirical measures approximately solve a kinetic transport equation, with a source term given by the flux of particles from both interior and boundary events. It is simpler to state the equations assuming that for each species k the weak limit $\mu_k(t) = \lim_{N\to\infty} \mu^N_k(t)$ has a number density

$$\mu_k(t)(dx) = f_k(x,t)\, dx. \tag{14}$$

We also define the total numbers of μ_k,

$$F_k(t) = \int_0^\infty f_k(x,t)\, dx, \quad F(t) = \sum_{k=1}^M F_k(t), \tag{15}$$

and the weighted fractions

$$W_k^{(l)}(t) = \frac{w_k^{(l)}}{\sum_{p=1}^M w_p^{(l)} F_p(t)}, \quad \gamma(t) = \frac{F(t)}{\sum_{p=1}^M w_p^{(0)} F_p(t)}. \tag{16}$$

Then for each species $\sigma \in \{1, \ldots, M\}$, the limiting density f_σ satisfies the kinetic transport equations

$$\partial_t f_\sigma(x,t) + v_\sigma \partial_x f_\sigma(x,t) = j_\sigma := j_\sigma^+(x,t) - j_\sigma^-(x,t), \tag{17}$$

$$j_\sigma^+(x,t) = \sum_{k=1}^M \left(\sum_{l=1}^{M_-} \dot{L}_l J_{\sigma k}^{(l)} W_k^{(l)}(t) + \beta\gamma(t) J_{\sigma k}^{(0)} w_k^{(0)} \right) f_k(x,t), \tag{18}$$

$$j_\sigma^-(x,t) = \left(\sum_{l=1}^{M_-} \dot{L}_l K^{(l)} W_\sigma^{(l)}(t) + \beta\gamma(t) K^{(0)} w_\sigma^{(0)} \right) f_\sigma(x,t), \tag{19}$$

$$\dot{L}_l = -f_l(0,t) v_l, \quad l = 1, \ldots, M_-. \tag{20}$$

While perhaps cumbersome at first sight, equation (17) is easily understood. The index σ denotes a fixed species under consideration. The left-hand side of (17) describes the advection of the number density f_σ under the constant velocity v_σ. The right-hand side describes the growth and loss of species σ due to fluxes $j_\sigma^\pm$ into and out of species σ. The fluxes are described in equations (18) and (19). In these equations, the index l enumerates all possible boundary events, and the index k enumerates all the species that could mutate to species σ. A boundary event for species $l \in S_-$ gives rise to both birth and death terms in proportion to the rate $\dot{L}_l = -f_l(0,t)v_l(0)$ and the weights $W_k^{(l)}(t) f_k(x,t)$. The weights $J_{k,\sigma}^{(l)}$ and $J_{k,\sigma}^{(0)}$ defined in equation (9) arise as we sum over all mutations that lead to the creation of particles of species σ of size x when a particle of species l hits the origin. Similarly, such particles may be lost when they are mutated. This occurs in proportion to the weight $W_\sigma^{(l)}(t)$. The terms multiplied by the rate $\beta\gamma(t)$ account for interior events.

The structure of the equations is more transparent when the flux j is written as a matrix vector product. Let $f = (f_1, \ldots, f_M)$ and $j = (j_1, \ldots, j_M)$. We may then write

$$j = \left(\sum_{l=1}^{l} A^{(l)} \dot{L}_l + \beta\gamma(t)\, A^{(0)} \right) f, \tag{21}$$

where the matrices $A^{(l)}$ and $A^{(0)}$ have off-diagonal terms given by

$$A_{\sigma k}^{(l)} = J_{\sigma k}^{(l)} W_k^{(l)}, \quad A_{\sigma k}^{(0)} = J_{\sigma k}^{(0)} w_k^{(0)}, \quad \sigma \neq k, \tag{22}$$

and diagonal terms given by

$$A_{\sigma\sigma}^{(l)} = -K^{(l)} W_\sigma^{(l)}, \quad A_{\sigma k}^{(0)} = -K_{\sigma k}^{(0)} w_\sigma^{(0)}. \tag{23}$$

The boundary values $f_l(0,t)$, for the incoming species $l \in S_-$ play a subtle role in the kinetic equation since they determine the rate of boundary events.

In order to obtain well-posedness of the kinetic equations, we will assume that the number densities are continuous on $[0,\infty)$ so that there is no ambiguity in defining their boundary values. In contrast, the boundary value of the outgoing species $l \in S_0, S_+$ do not affect the flux and we impose the boundary conditions

$$f_l(0,t) = 0, \quad l = M_- + 1, \ldots, M. \tag{24}$$

2.3 The hydrodynamic limit theorem

We consider the class of test functions[2]

$$\mathcal{C} = \{\psi \in C^1(\mathbb{R}_+) \cap C_b(\mathbb{R}_+) : \psi' \in C_b(\mathbb{R}_+)\}, \tag{25}$$

and pair empirical measures with test functions in the natural manner

$$\langle \mu_\sigma^N, \psi \rangle = \int_{\mathbb{R}_+} \psi(x) \mu_\sigma^N(dx) = \frac{1}{N} \sum_{i=1}^{N(t)} \psi(x_i) \mathbf{1}_{\{s_i = \sigma\}}. \tag{26}$$

Our main theorem describes the convergence of empirical measures for the particle process to a weak form of the limiting PDE (17), assuming convergence of the initial empirical measures to a nonatomic measure.

Theorem 1. *Assume the initial empirical measures $\mu_\sigma^{0,N}$ for the particle process converge weakly to measures μ_σ^0 in $\mathcal{M}(\mathbb{R}_+)$ as $N \to \infty$ for each species $\sigma \in \{1, \ldots, M\}$. Assume further that the measures μ_σ^0 are non-atomic and that $\mu_\sigma^0(\mathbb{R}_+) > 0$ for $\sigma \in S_-$. Then*

1. *There exists a time $T_e > 0$ such for every species σ the empirical measures $\mu_\sigma^N(t)$ converge along a subsequence to limits $\mu_\sigma(t)$ in the Skorohod topology $\mathbb{D}([0,T_e], \mathcal{M}(\mathbb{R}_+))$.*

2. *For each species σ, the normalized number of particles lost $L_\sigma^N(t)$ defined in (56), converges along a subsequence to a continuous cumulative distribution function $L_\sigma(t)$ in the mean L^∞ metric:*

$$\lim_{N\to\infty} \mathbb{E}\left[\sup_{t \le T_e} \left| L_\sigma^N(t) - L_\sigma(t) \right|\right] = 0, \quad 1 \le \sigma \le M. \tag{27}$$

3. *The limiting measures $\mu_\sigma(t)$ and $L_\sigma(t)$ for each species σ satisfy the following weak form of the kinetic equation (17). For every $\psi \in \mathcal{C}$*

$$\begin{aligned} \langle \mu_\sigma(t), \psi \rangle + L_\sigma(t)\psi(0) = \langle \mu_\sigma^0, \psi \rangle + \int_0^t \langle \mu_\sigma(s), \psi' v_\sigma \rangle + \\ \sum_{l=1}^{M_-} \int_0^t \left(-K^{(l)} W_\sigma^{(l)}(s) \langle \mu_\sigma(s), \psi \rangle + \sum_{k=1}^{M} J_{k,\sigma}^{(l)} W_k^{(l)}(s) \langle \mu_k(s), \psi \rangle \right) dL_l(s) \\ + \beta \int_0^t \gamma(s) \left(-K^{(0)} w_\sigma^{(0)} \langle \mu_\sigma(s), \psi \rangle + \sum_{k=1}^{M} J_{k,\sigma}^{(0)} w_k^{(0)} \ \langle \mu_k(s), \psi \rangle \right) ds. \end{aligned} \tag{28}$$

[2] GM. does not assume $\psi(0) = 0$.

Remark 1. *The Skorohod topology is reviewed in Section 3.4. Since individual jumps of the prelimit PDMP converge to zero as $N \to \infty$, the limiting functions $\langle \mu_k(t), \psi \rangle, k = 1, \ldots, M$ are continuous in t, meaning that empirical measures also converge in the local uniform metric (see Prop. 1.17 in Chapter VI of [10]).*

Remark 2. *The proof of Theorem relies on the construction of suitable martingales for the M-species model and tightness estimates. These issues are discussed in Section 3.5 below after we review the theory of PDMPs.*

Remark 3. *As we will see in Section* **??** *for continuous solutions the loss measures have a density $L_\sigma(t) = -v_\sigma(0) f_\sigma(0,t)dt$.*

2.4 Well-posedness of the kinetic equations

We also state a well-posedness theorem for $L^1 \cap L^\infty$ initial data, which is proved in Section 5.

Definition 1. *For $k = 1, \ldots, M$, let $u_k(x,t) \in C([0,T_e], L^1 \cap L^\infty(\mathbb{R}_+, \mathbb{R}_+))$. We call $u_k(x,t)$ a* ***weak solution in*** *$L^1 \cap L^\infty(\mathbb{R}_+, \mathbb{R}_+)$ of (**??**) with initial conditions $u_k^0(x) \in L^1 \cap L^\infty(\mathbb{R}_+, \mathbb{R}_+)$, if for all $\psi \in \mathcal{C}$, there exist $\mu_k(t)$, $t \in [0, T_e]$, that satisfy (**??**), with $\mu_k(t)(dx) = u_k(x,t)dx$, and $\mu_k(0)(dx) = u_k^0(x)dx$.*

Theorem 2. *For $k = 1, \ldots, M$, there exists a unique weak solution $u_k(x,t)$ in $L^1 \cap L^\infty(\mathbb{R}_+, \mathbb{R}_+)$ of (**??**) with initial conditions $u_k^0(x) \in L^1 \cap L^\infty(\mathbb{R}_+, \mathbb{R}_+)$.*

Remark 4. *Theorem 2 should not be mistaken as stating that empirical measures with $L^1 \cap L^\infty$ initial data converge in the uniform metric to a unique $L^1 \cap L^\infty$ solution of (**??**). Rather, in the class of all possible solutions, there is one, and only one, solution that is in $C([0,T_e], L^1 \cap L^\infty(\mathbb{R}^+))$*

Well-posedness of (**??**) for L^1 data is discussed in Section 5.

3 The M-species model as a PDMP

We briefly review the basics of PDMPs, following Davis [3], and then explain how the M-species particle process of Section 2.1 fits into this framework.

3.1 Background: General theory of PDMPs

We consider a countable set $\mathcal{S}$ with elements denoted $\mathbf{s}$, a map $\mathbf{d} : \mathcal{S} \to \mathbb{N}$, and open sets for each $\mathbf{s}$ of the form $M_\mathbf{s} \subset \mathbb{R}^{\mathbf{d}(\mathbf{s})}$. The state space is the disjoint union

$$E = \coprod_{\mathbf{s} \in \mathcal{S}} M_\mathbf{s} = \{(\mathbf{s}, \mathbf{x}) : \mathbf{s} \in \mathcal{S}, \mathbf{x} \in M_\mathbf{s}\}. \tag{29}$$

The space E has a natural topology. Let $\iota_\mathbf{s} : M_\mathbf{s} \to E$ be the canonical injection defined by $\iota_\mathbf{s}(\mathbf{x}) = (\mathbf{s}, \mathbf{x})$. A set $A \subset E$ is open if for every $\mathbf{s}$, $\iota_\mathbf{s}^{-1}(A)$ is open in $M_\mathbf{s}$. The collection of all open sets may be used to define the set $\mathcal{E}$ of Borel subsets of E. This makes $(E, \mathcal{E})$ a Borel space.

A PDMP is an E-valued generalized jump process $X(t) = (\mathbf{s}(t), \mathbf{x}(t))$, $t \geq 0$, that is prescribed by:

1. Sufficiently smooth vector fields $\mathbf{v_s} : M_\mathbf{s} \to \mathbb{R}^{\mathbf{d(s)}}$, $\mathbf{s} \in \mathcal{S}$.
2. A measurable function $\lambda : E \to \mathbb{R}^+$.
3. A transition measure $Q : \mathcal{E} \times (E \cup \Gamma^*) \to [0, 1]$. Here Γ^* denotes the exit boundary defined in equations (32)–(31) below.

Points in $M_\mathbf{s}$ travel according to flows defined by the vector fields $\mathbf{v_s}$ until either a Poisson clock with intensity $\lambda(\mathbf{s}, \mathbf{x})$ rings or the point $\mathbf{x}(t)$ hits the exit boundary Γ^*. When such a *critical event* occurs the point $X(t)$ jumps to a random new position whose law is given by Q.

Each vector field $\mathbf{v_s}$ may be viewed as a first-order differential operator on $M_\mathbf{s}$. We assume they define a flow $\varphi_\mathbf{s}(t, \mathbf{x})$ such that

$$\frac{\partial}{\partial t} h\left(\mathbf{s}, \varphi_\mathbf{s}(t, \mathbf{x})\right) = \mathbf{v_s}\left(h\left(\varphi_\mathbf{s}(t, \mathbf{x})\right)\right), \quad \varphi_\mathbf{s}(0, \mathbf{x}) = \mathbf{x}, \tag{30}$$

for all sufficiently smooth test functions h and for t in a maximal interval of existence. The flow terminates only when $\mathbf{x}(t)$ hits

$$\partial^* M_\mathbf{s} = \left\{\mathbf{y} \in \partial M_\mathbf{s} : \varphi_\mathbf{s}(t^-, \mathbf{x}) = \mathbf{y} \quad \text{for some } (t, \mathbf{x}) \in \mathbb{R}_+ \times M_\mathbf{s}\right\}. \tag{31}$$

The *exit boundary* is the disjoint collection

$$\Gamma^* = \coprod_{\mathbf{s} \in \mathcal{S}} \partial^* M_\mathbf{s} = \left\{(\mathbf{s}, \mathbf{x}) : \mathbf{s} \in \mathcal{S}, \mathbf{x} \in \partial^* M_\mathbf{s}\right\}, \tag{32}$$

At a given state $(\mathbf{s}, \mathbf{x}) \in E$ we define the first exit time

$$t^*_\mathbf{s}(\mathbf{x}) = \sup\{t > 0 : \varphi_\mathbf{s}(t, \mathbf{x}) \in M_\mathbf{s}\}, \tag{33}$$

and the survivor function

$$\mathcal{F}_{(\mathbf{s},\mathbf{x})}(t) = \begin{cases} \exp\left(-\int_0^t \lambda\left(\mathbf{s}, \varphi_\mathbf{s}\left(\tau, \mathbf{x}\right)\right) d\tau\right), & t < t^*_\mathbf{s}(\mathbf{x}), \\ 0, & t \geq t^*(\mathbf{x}). \end{cases} \tag{34}$$

The stochastic process $(X(t))_{t \geq 0}$ with initial condition $X(0) = (\mathbf{s}_0, \mathbf{x}_0)$ is defined as follows. Choose a random time T_0 such that $\mathbb{P}[T_0 > t] = \mathcal{F}_{(\mathbf{s}_0,\mathbf{x}_0)}(t)$ and an E-valued random variable $(\mathbf{s}_1, \mathbf{x}_1)$ with law $Q(\cdot\,; \varphi_{\mathbf{s}_0}(T_0, \mathbf{x}_0))$ that is independent of T_0. The trajectory of $X(t)$ for $t \leq T_0$ is then

$$X(t) = \begin{cases} \left(\mathbf{s}_0, \varphi_{\mathbf{s}_0}(t, \mathbf{x}_0)\right), & t < T_0, \\ (\mathbf{s}_1, \mathbf{x}_1), & t = T_0. \end{cases} \tag{35}$$

At $t = T_0$, we repeat this process, replacing the jump time T_0 in the algorithm above with $T_1 - T_0$ and the state $(\mathbf{s}_0, \mathbf{x}_0)$ with $(\mathbf{s}_1, \mathbf{x}_1)$. Iterating this process, jump by jump, yields a cadlag process $X(t)$, $t \in [0, \infty)$.

Under modest assumptions, it can be shown that $X(t)_{t\geq 0}$ is a strong Markov process [3, §3]. We only require that $Q\left(A;(\mathbf{s},\mathbf{x})\right)$ is a measurable function of $(\mathbf{s},\mathbf{x})$ for each Borel set $A \in \mathcal{E}$ and a probability measure on $(E,\mathcal{E})$ for each $(\mathbf{s},\mathbf{x}) \in E \cup \Gamma^*$. The rate function $\lambda : E \to \mathbb{R}^+$ must be measurable with a little integrability: specifically, for each state $(\mathbf{s},\mathbf{x}) \in E$ we require the existence of $\varepsilon > 0$ such that the function $\tau \to \lambda(\mathbf{s}, \varphi_{\mathbf{s}}(\tau,\mathbf{x}))$ is summable for $\tau \in [0,\varepsilon)$. These conditions are easily verified in our model.

3.2 The generator and martingales

The PDMP has an associated infinitesimal generator $\mathcal{A}$. The domain $\mathcal{D}(\mathcal{A})$ of $\mathcal{A}$ is the set of test functions $h : E \to \mathbb{R}$ for which the limit

$$\mathcal{A}h(\mathbf{s},\mathbf{x}) = \lim_{t\to 0^+} \frac{1}{t}\left(\mathbb{E}^{(\mathbf{s},\mathbf{x})}(h(X(t))) - h(\mathbf{s},\mathbf{x})\right) \tag{36}$$

exists for all $(\mathbf{s},\mathbf{x}) \in E$. For $h \in \mathcal{D}(\mathcal{A})$

$$\mathcal{A}h(\mathbf{s},\mathbf{x}) = \mathbf{v}_{\mathbf{s}}(h(\mathbf{x})) + \lambda(\mathbf{s},\mathbf{x}) \int_E (h(\mathbf{r},\mathbf{y}) - h(\mathbf{s},\mathbf{x}))Q(d(\mathbf{r},\mathbf{y});\mathbf{s},\mathbf{x}). \tag{37}$$

Since E is a countable disjoint union of open sets $M_{\mathbf{s}} \subset \mathbb{R}^{\mathbf{d}(\mathbf{s})}$, Q is a product of counting measure on $\mathcal{S}$ and a measure $P_{\mathbf{r}}$ on $M_{\mathbf{r}}$, $\mathbf{r} \in \mathcal{S}$, of the form

$$\int_E h(\mathbf{r},\mathbf{y})Q(d(\mathbf{r},\mathbf{y});\mathbf{s},\mathbf{x}) = \sum_{\mathbf{r}\in\mathcal{S}} \int_{M_{\mathbf{r}}} h(\mathbf{r},\mathbf{y})P_{\mathbf{r}}(d\mathbf{y};\mathbf{s},\mathbf{x}). \tag{38}$$

Finally, by Dynkin's formula, each $h \in \mathcal{D}(\mathcal{A})$ gives rise to the martingale

$$M_t^h := h(X(t)) - h(X(0)) - \int_0^t \mathcal{A}h(X(\tau))\, d\tau. \tag{39}$$

While the above definitions are in accordance with the usual theory of Markov process, it is somewhat subtle to determine when a test function lies in $\mathcal{D}(\mathcal{A})$. We will use the following sufficient conditions [15]

Theorem 3. *A function h is in $\mathcal{D}(\mathcal{A})$ if all of the following conditions hold:*

1. *The function $t \mapsto h(\varphi_{\mathbf{s}}(t,\mathbf{x}))$ is absolutely continuous on $[0, t^*_{\mathbf{s}}(x))$ for every $(\mathbf{s},\mathbf{x}) \in E$.*
2. *$\lim_{t\to 0} h\left(\varphi_{\mathbf{s}}(t,\mathbf{x})\right) = h(\mathbf{s},\mathbf{x})$ for each $(\mathbf{s},\mathbf{x}) \in \Gamma^*$.*
3. *Boundary conditions:*

$$h(\mathbf{s},\mathbf{x}) = \int_E h(\mathbf{r},\mathbf{y})Q\left(d(\mathbf{r},\mathbf{y});\mathbf{s},\mathbf{x}\right), \quad (\mathbf{s},\mathbf{x}) \in \Gamma^*. \tag{40}$$

4. *Finite variation of jumps: For all $t \in \mathbb{R}^+$ and $(\mathbf{s},\mathbf{x}) \in E$,*

$$\mathbb{E}^{(\mathbf{s},\mathbf{x})}\left(\sum_{m=1}^{\infty} \mathbf{1}_{T_m<t}\ |h(X(T_m) - h(X(T_{m-1}))|\right) < \infty. \tag{41}$$

3.3 The M-species model as a PDMP

We now show the M-species model defined in Section 2.1 is a PDMP. Define the countable set of *species indices*

$$\mathcal{S} = \bigcup_{m\in\mathbb{N}} \{1,\dots,M\}^m. \tag{42}$$

It is convenient to introduce notation that makes explicit the distinction between the number of particles in a state $(\mathbf{s},\mathbf{x})$ and the fixed parameter N that is the normalizing factor in the empirical measure (13). We denote the number of particles in the state $(\mathbf{s},\mathbf{x})$ by $|\mathbf{s}|$ and write

$$(\mathbf{s},\mathbf{x}) = \left(s_1,\dots,s_{|\mathbf{s}|};x_1,\dots,x_{|\mathbf{s}|}\right), \tag{43}$$

and the associated empirical measures is

$$\mu_\sigma^N(\mathbf{s},\mathbf{x}) = \frac{1}{N}\sum_{i=1}^{|\mathbf{s}|}\delta_{x_i}\mathbf{1}_{s_i=\sigma}, \quad \sigma = 1,\dots,M. \tag{44}$$

For the M-species process, $N(t) = |\mathbf{s}|(t)$, and equations (2),(13) and equations (43)–(44)are consistent. Similarly, each open set $M_{\mathbf{s}} = \mathbb{R}_+^{|\mathbf{s}|}$ and

$$E = \coprod_{\mathbf{s}\in\mathcal{S}} \mathbb{R}_+^{|\mathbf{s}|} = \left\{(\mathbf{s},\mathbf{x}) : \mathbf{s}\in\mathcal{S},\, \mathbf{x}\in\mathbb{R}_+^{|\mathbf{s}|}\right\}. \tag{45}$$

The velocity fields $\mathbf{v_s}$ on E are obtained from the velocity fields v_s, $s = 1,\dots,M$ of the M-species model,

$$\mathbf{v_s} = \sum_{i=1}^{|\mathbf{s}|} v_{s_i}(x_i)\frac{\partial}{\partial x_i}, \tag{46}$$

and the exit boundary is

$$\Gamma^* = \{(\mathbf{s},\mathbf{x})\in E| \text{ there exists } (s_i,x_i) \text{ such that } x_i = 0,\, s_i\in S_-\}. \tag{47}$$

In order to define the transition kernel Q, we first describe the finite set of 'neighbors' $E^\partial_{\mathbf{s},\mathbf{x}}$ for each state $(\mathbf{s},\mathbf{x})\in\Gamma^*$. Each point $(\mathbf{s},\mathbf{x})$ has a finite number, p, of particles with size zero. Let us label these particles with indices $i = k_1$,k_2,..., k_p, ordered such that the species $s_{k_1}\le s_{k_2}\le\dots s_{k_p}$. Let us begin by discussing the case when $p = 1$ (this is the most important case, since boundary events happen at distinct times with probability 1). When $p = 1$, the set $E^\partial_{\mathbf{s},\mathbf{x}}$ may be decomposed into M_- subsets, corresponding to boundary events at M_- species. More precisely, a boundary event occurs at species l, if the size $x_{j_1} = 0$ and the associated species $s_{j_1} = l$. According to the rules of Section 2.1, at such a boundary event, $K^{(l)}$ random variables (S_j,X_j) are chosen, and mutated as in equation (6). Each such mutation gives rise to a neighbor $(\mathbf{r},\mathbf{y})$ of $(\mathbf{s},\mathbf{x})$. Since the X_j are a random collection of $K^{(l)}$ points of $\mathbf{x}$, we may write $X_j = x_{i_j}$, for indices $i_1,\dots,i_{K^{(l)}}$. Then $(\mathbf{r},\mathbf{y})$ is obtained from $(\mathbf{s},\mathbf{x})$ in two 'sub-steps':

(i) *Pure mutation:* $\mathbf{x}$ is unchanged. The coordinates of $\mathbf{s}$ are changed as follows: $s_{i_j} \mapsto R^{(l)}_j$, $j = 1, \ldots, K^{(l)}$. Call this intermediate state $\hat{\mathbf{s}}$.

(ii) *Removal of zero size:* $\mathbf{x}$ is changed to $\mathbf{y}$ by deleting the particle x_{j_1} with size zero.

The probability $p^{\partial}(\cdot; \mathbf{s}, \mathbf{x})$ of each transition $(\mathbf{s}, \mathbf{x}) \mapsto (\mathbf{r}, \mathbf{y}) \in E^{\partial}_{\mathbf{s},\mathbf{x}}$ is given by the rules of Section 2.1. Finally, observe that these rules extend naturally to degenerate boundary points, where $0 = x_{k_1} = x_{k_2} = \ldots x_{k_p}$. In this case, according to the rules of Section 2.1, we order the points $x_{j_1}, \ldots, x_{j_p}$ so that the species $s_{j_1} < s_{j_2} < s_{j_p}$, and mutate and remove particles p times in sequence as above.

Similarly, given an interior point $(\mathbf{s}, \mathbf{x}) \in E$ we can use the mutation matrix $R^{(0)}$ and the weights $w^{(0)}$ to define a set of interior points $E^{(0)}_{\mathbf{s},\mathbf{x}}$ that $(\mathbf{s}, \mathbf{x})$ jumps to along with the corresponding probabilities $p^{(0)}(\cdot; \mathbf{s}, \mathbf{x})$. In this case, the transition involves only a mutation and no removal of zero sizes.

In summary, the transition kernel is given by

$$Q(A; \mathbf{s}, \mathbf{x}) = \begin{cases} \int_A p^{\partial}(\mathbf{r}, \mathbf{y}; \mathbf{s}, \mathbf{x}) \mathbf{1}_{E^{\partial}_{\mathbf{s},\mathbf{x}}}(\mathbf{r}, \mathbf{y})\, d(\mathbf{r}, \mathbf{y}), & (\mathbf{s}, \mathbf{x}) \in \Gamma^*, \\ \int_A p^{(0)}(\mathbf{r}, \mathbf{y}; \mathbf{s}, \mathbf{x}) \mathbf{1}_{E^{(0)}_{\mathbf{s},\mathbf{x}}}(\mathbf{r}, \mathbf{y})\, d(\mathbf{r}, \mathbf{y}), & (\mathbf{s}, \mathbf{x}) \in E. \end{cases} \tag{48}$$

Since each particle carries an independent Poisson-β clock β, the first time T that a clock rings follows the distribution $T \sim \min_{1 \le i \le |\mathbf{s}|} \text{Poisson}(\beta) = \text{Poisson}(|\mathbf{s}|\beta)$. Thus

$$\lambda(\mathbf{s}, \mathbf{x}) = \beta |\mathbf{s}|. \tag{49}$$

This completes the description of the M-species model as a PDMP.

3.4 Tightness in the Skorokhod topology

The most technical aspects of the proof of Theorem 1 involve compactness of stochastic processes. We summarize the facts we need here.

Let $(\mathfrak{M}, d)$ denote a metric space with metric d. For our purposes, $\mathfrak{M}$ will either be $\mathbb{R}_+$ or $\mathcal{M}(\mathbb{R}_+)$, the space of finite measures on $\mathbb{R}_+$ with the Prohorov metric [2]. We use the Skorohod $J1$ topology, denoted $\mathbb{D}([0,t], \mathfrak{M})$. The $J1$ topology allows for convergence of functions that "wiggle" in time as well as space, and has the following characterization (see [10]):

Let $\mathscr{R}$ be the set of continuous functions $r : [0,t] \to [0,t]$ that are strictly increasing and satisfy $r(0) = 0$ and $r(t) = t$. These functions are *reparametrizations* of the time interval $[0,t]$.

Definition 2. *A sequence of functions $\alpha_n \to \alpha$ in $\mathbb{D}([0,t], \mathfrak{M})$ if and only if there is a sequence $r_n \in \mathscr{R}$ such that*

1. $\sup_s |r_n(s) - s| \to 0$, *and*
2. $\sup_{s \le t} d(\alpha_n(r_n(s)), \alpha(s)) \to 0$, *as $n \to \infty$.*

If $\alpha(t)$ is continuous, the functions actually converge in the local uniform topology. In general, however, the local uniform topology is strictly stronger than the Skorohod J1 topology. The following criterion are used to establish compactness of empirical measures.

Lemma 1. *Let $T > 0$. A sequence $\nu_k(t)$ in $\mathbb{D}([0,T], \mathcal{M}(\mathbb{R}^+))$ is tight if and only if the sequence of functions $\langle \phi, \nu_k(t) \rangle$ is tight in $\mathbb{D}([0,T], \mathbb{R}^+)$ for all $\phi \in C_b(\mathbb{R}^+)$.*

Theorem 4. ***(Aldous' conditions)*** *Let $T > 0$. A sequence of stochastic processes $X_n(t)$ is tight in $\mathbb{D}([0,T], \mathbb{R}^+)$ if the following two conditions hold:*

1. *For all rational $t \in [0,T)$ and for all $\varepsilon > 0$, there exists an $L > 0$ such that*
$$\sup_{n>0} \mathbb{P}(|X_n(t)| > L) \leq \varepsilon. \tag{50}$$
2. *For jump times $\tau_1, \ldots, \tau_{m(t)}$, then for all $\varepsilon > 0$,*
$$\lim_{r \to 0} \limsup_{n \to \infty} \sup_{s < r, \tau_i} \mathbb{P}(|X_n((\tau_i + s) \wedge T) - X_n(\tau_i)| > \varepsilon) = 0. \tag{51}$$

3.5 Empirical measures and test functions.

The proof of Theorem 1 relies on the construction of martingales for the M-species PDMP that approximate equation (28). Since each test function in $\mathcal{D}(\mathcal{A})$ generates a martingale, it is natural to consider a test function $h : E \to \mathbb{R}$ obtained by pairing the empirical measure μ_σ defined in (44) with an arbitrary $\psi \in \mathcal{C}$ as follows

$$h(\mathbf{s}, \mathbf{x}) = \langle \mu_\sigma, \psi \rangle = \frac{1}{N} \sum_{i=1}^{|\mathbf{s}|} \psi(x_i) \mathbf{1}_{s_i = \sigma}. \tag{52}$$

However, in general, such a test function does *not* lie in $\mathcal{D}(\mathcal{A})$!

This difficulty arises because the empirical measure μ_σ keeps track of only species σ, while mutations couple the sizes of all species together. In order that $h \in \mathcal{D}(\mathcal{A})$, it must satisfy the boundary condition (3) in Theorem 3. In terms of the transition measure (48), this boundary condition is

$$\sum_{(\mathbf{r}, \mathbf{y}) \in E^\partial_{\mathbf{s}, \mathbf{x}}} p^\partial(\mathbf{r}, \mathbf{y}; \mathbf{s}, \mathbf{x}) \left(h(\mathbf{r}, \mathbf{y}) - h(\mathbf{s}, \mathbf{x}) \right) = 0. \tag{53}$$

The sum over neighbors $(\mathbf{r}, \mathbf{y}) \in E^\partial_{\mathbf{s}, \mathbf{x}}$ may be decomposed into a sum over all boundary events. At a boundary event at species l, each neighbor $(\mathbf{r}, \mathbf{y})$ differs from $(\mathbf{s}, \mathbf{x})$ by a pure mutation and removal of zero sizes as described in Section 3.3 above. Let $i_1, \ldots, i_{K^{(l)}}$ denote the indices at which $\mathbf{s}$ differs from $\hat{\mathbf{s}}$. We then find

$$h(\mathbf{r}, \mathbf{y}) - h(\mathbf{s}, \mathbf{x}) = \frac{1}{N} \sum_{j=1}^{K^{(l)}} \psi(x_{i_j}) \left(\mathbf{1}_{R^{(l)}_{s_{i_j}, j} = \sigma} - \mathbf{1}_{s_{i_j} = \sigma} \right). \tag{54}$$

Thus, (53) gives a complicated system of functional equations that should not be expected to hold for arbitrary $\psi \in \mathcal{C}$.

3.6 The loss measure $L(t)$ and mutation measure $\nu(t)$

In order to use test functions akin to (52), it is necessary to augment the state space of the M-species model to include an atom at the origin that accounts for the vanishing of particles and an $M \times M$ matrix of measures ν that keeps track of all mutations. It is necessary to introduce some notation to define these measures properly.

The jump times of $X(t)$ have been denoted $\{T_m\}_{m=1}^{\infty}$. Each of these jumps corresponds to one of M_- possible boundary events, or an interior event. We partition the sequence of jump times $\{T_m\}_{m=1}^{\infty}$ into $M_- + 1$ ordered subsequences corresponding to each of these possibilities. Let $\tau_1^{(l)}, \tau_2^{(l)}, \ldots$ denote the ordered subsequence of jump times corresponding to boundary events at species l. Similarly, let $\tau_1^{(0)}, \tau_2^{(0)}, \ldots$ denote the ordered subsequence of jump times corresponding to interior events.

The normalized number of particles of species l lost upto time t, and the total number of particles lost upto time t, are given respectively by

$$L_l(t) := \frac{1}{N} \sum_{m=1}^{\infty} \mathbf{1}_{\{\tau_{\mathbf{m}}^{(\mathbf{l})} < \mathbf{t}\}}, \quad L(t) = \sum_{l=1}^{M_-} L_l(t). \tag{55}$$

The derivatives of these functions are the positive measures

$$\dot{L}_l(t) = \frac{1}{N} \sum_{m=1}^{\infty} \delta_{\tau_m^{(l)}}(dt), \quad \dot{L}(t) = \sum_{l=1}^{M_-} \dot{L}_l(t). \tag{56}$$

We augment the empirical measure with an atom at the origin to account for the loss of species l by defining the *loss measures* $L_l(t)\delta_0$, $l = 1, \ldots, M_-$. It is also convenient to define $L_l(t) \equiv 0$ for $l \in S_0$ and $l \in S_+$ in order to state the governing equations for all species together (as in equation (28)).

At each jump time $\tau_m^{(l)}$, the mutation is determined by the random integers $S_{1,m}^{(l)}, \ldots, S_{K^{(l)},m}^{(l)}$ and the random sizes $X_{1,m}^{(l)}, \ldots, X_{K^{(l)},m}^{(l)}$ that select the $K^{(l)}$ particles being mutated (the notation here augments that of Section 2.1 to include the l and m dependence). Thus, we have a mutation from species ρ to species σ at time $\tau_m^{(l)}$ when $S_{j,m}^{(l)} = \rho$ and $R_{S_{j,m}^{(l)},j}^{(l)} = \sigma$ for some j.

The (ρ, σ) entry of the *mutation measure*, $\nu_{\rho,\sigma}(t)$, keeps track of all mutations from species ρ to species σ before time t. It is defined by

$$\nu_{\rho,\sigma}(t) = \frac{1}{N} \sum_{m=1}^{\infty} \sum_{l=1}^{M} \sum_{j=1}^{K^{(l)}} \mathbf{1}_{\{\tau_m^{(l)} < t\}} \mathbf{1}_{\{S_{j,m}^{(l)} = \rho\}} \mathbf{1}_{\{R_{S_{j,m}^{(l)},j}^{(l)} = \sigma\}} \delta_{X_{j,m}^{(l)}}. \tag{57}$$

The sum over m in equations (56) and (57) is finite with probability 1 as may be seen in the following estimate.

Lemma 2. *Consider the PDMP* $(\mathbf{s},\mathbf{x})(t)$, $t\geq 0$ *and assume* $|\mathbf{s}(0)|=N$. *Then for each* $t>0$

$$\mathbb{E}\left(\sum_{m=1}^{\infty}\mathbf{1}_{\{\tau_m^{(0)}<t\}}\right) \leq N\beta t, \tag{58}$$

$$\mathbb{E}\left(\sum_{m=1}^{\infty}\sum_{l=1}^{M}\mathbf{1}_{\{\tau_m^{(l)}<t\}}\right) \leq N, \tag{59}$$

$$\mathbb{E}\left(\sum_{m=1}^{\infty}\mathbf{1}_{\{T_m<t\}}\right) \leq N\left(\beta t+1\right). \tag{60}$$

Proof. There are at most N particles in the system, each carrying a Poisson β clock. Thus, the rate of the Poisson process $\tau_m^{(0)}$ is at most $N\beta$. This proves (58). As for (59), there are at most N particles, thus there can be at most N boundary events in any interval $[0,t]$. Finally, (60) follows from summing (58) and (59). □

3.7 The extended PDMP and its transition kernel

The extended state space is

$$\begin{aligned}\tilde{E} &= \coprod_{\mathbf{s}\in\mathcal{S}}\mathbb{R}_+^{|\mathbf{s}|}\times\mathcal{M}(\mathbb{R}_+)^{M\times M}\times\mathcal{M}(\mathbb{R}_+)^M \\ &= \left\{(\mathbf{s},\mathbf{x},\nu,L\delta_0):\mathbf{s}\in\mathcal{S},\,\mathbf{x}\in\mathbb{R}_+^{|\mathbf{s}|},\nu\in\mathcal{M}(\mathbb{R}_+)^{M\times M},L\in\mathbb{R}^M\right\}.\end{aligned} \tag{61}$$

States in $\tilde{E}$ are denoted $(\mathbf{s},\tilde{\mathbf{x}}):=(\mathbf{s},\mathbf{x},\nu,L\delta_0)$ and the associated stochastic process $\tilde{X}(t)=(\mathbf{s},\tilde{\mathbf{x}})(t)$. The set $\mathcal{M}(\mathbb{R}_+)$ equipped with the Prohorov metric is a complete, separable metric space. Thus, the open sets and Borel sets of $\tilde{E}$ may be defined as in Section 3.1. We denote the associated Borel space $\tilde{\mathcal{E}}$.

In order to construct a well-defined PDMP on $(\tilde{E},\tilde{\mathcal{E}})$ we must specify the extended velocity field $\tilde{\mathbf{v}}_{\mathbf{s}}$, rates of transition $\tilde{\lambda}(\mathbf{s},\tilde{\mathbf{x}})$ and transition kernel $\tilde{Q}$ as in Section 3.3. We extend the velocity field so that ν has no drift

$$\tilde{\mathbf{v}}(\mathbf{s},\tilde{\mathbf{x}})=(\mathbf{v}_{\mathbf{s}}(\mathbf{x}),0). \tag{62}$$

As a consequence, the extended exit boundary is simply

$$\tilde{\Gamma}^*=\{(\mathbf{s},\tilde{\mathbf{x}})\,|(\mathbf{s},\mathbf{x})\in\Gamma^*\}. \tag{63}$$

In order to define the extended transition kernel $\tilde{Q}$ on $\tilde{E}\times\tilde{\Gamma}^*$, we must consider all possible interior and boundary events and their probabilities. The set of extended interior events is

$$\tilde{E}^{(0)}_{\mathbf{s},\tilde{\mathbf{x}}}=\left\{(\mathbf{r},\tilde{\mathbf{y}})\,\middle|(\mathbf{r},\mathbf{y})\in E^{(0)}_{\mathbf{s},\mathbf{x}}\right\}. \tag{64}$$

The corresponding transition probabilities and rate are

$$\tilde{p}^{(0)}(\mathbf{r},\tilde{\mathbf{y}};\mathbf{s},\tilde{\mathbf{x}})=p^{(0)}(\mathbf{r},\mathbf{y};\mathbf{s},\mathbf{x}),\quad \tilde{\lambda}(\mathbf{s},\tilde{\mathbf{x}})=\lambda(\mathbf{s},\mathbf{x}). \tag{65}$$

The loss and mutation measure changes only at boundary events. Suppose the extended system is in state $(\mathbf{s}, \tilde{\mathbf{x}})$ and a particle of species l hits the origin. Then the normalized number changes by

$$\triangle L_k = \frac{1}{N}\mathbf{1}_{k=l}, \quad k = 1, \ldots, M. \tag{66}$$

The mutation measure changes as follows. We know that the state of the (unextended) system $(\mathbf{s}, \mathbf{x})$ jumps to a state $(\mathbf{r}, \mathbf{y}) \in E^{\partial}(\mathbf{s}, \mathbf{x})$ obtained via a choice of species $S_1, \ldots, S_{K^{(l)}}$ and sizes $X_1, \ldots, X_{K^{(l)}}$ as described in Section 3.3. It follows that the ρ, σ entry of the mutation measure jumps by (notation as in equation (54))

$$\triangle \nu_{\rho,\sigma} = \frac{1}{N}\sum_{j=1}^{K^{(l)}} \delta_{x_{i_j}} \mathbf{1}_{s_{i_j}=\rho} \mathbf{1}_{R^{(l)}_{s_{i_j},j}=\sigma}. \tag{67}$$

With $\triangle\nu$ defined as above, we find

$$\tilde{E}^{\partial}_{\mathbf{s},\tilde{\mathbf{x}}} = \left\{ (\mathbf{r}, \mathbf{y}, \nu + \triangle\nu, (L + \triangle L)\delta_0) \,\middle|\, (\mathbf{r}, \mathbf{y}) \in E^{\partial}_{\mathbf{s},\mathbf{x}} \right\}. \tag{68}$$

At degenerate boundary events where p of the x-coordinates of $(\mathbf{x}, \mathbf{s}) \in \Gamma^*$ vanish, the change in the mutation measure is the sum of p signed measures of the form above.

Thus, every state $(\mathbf{r}, \tilde{\mathbf{y}}) \in \tilde{E}^{\partial}_{\mathbf{s},\tilde{\mathbf{x}}}$ is obtained from a state $(\mathbf{r}, \mathbf{y}) \in E^{\partial}_{\mathbf{s},\mathbf{x}}$. The transition measure $\tilde{p}^{\partial}(\mathbf{r}, \tilde{\mathbf{y}}; \mathbf{s}, \tilde{\mathbf{x}})$ is obtained by pushing forward $p^{\partial}(\mathbf{r}, \mathbf{y}; \mathbf{s}, \mathbf{x})$ under this map. In summary,

$$\tilde{Q}(\tilde{A}; \mathbf{s}, \tilde{\mathbf{x}}) = \begin{cases} \int_{\tilde{A}} \tilde{p}^{\partial}(\mathbf{r}, \tilde{\mathbf{y}}; \mathbf{s}, \tilde{\mathbf{x}}) \mathbf{1}_{\tilde{E}^{\partial}_{\mathbf{s},\tilde{\mathbf{x}}}}(\mathbf{r}, \tilde{\mathbf{y}})\, d(\mathbf{r}, \tilde{\mathbf{y}}), & (\mathbf{s}, \tilde{\mathbf{x}}) \in \tilde{\Gamma}^*, \\ \int_{\tilde{A}} \tilde{p}^{(0)}(\mathbf{r}, \tilde{\mathbf{y}}; \mathbf{s}, \tilde{\mathbf{x}}) \mathbf{1}_{\tilde{E}^{(0)}_{\mathbf{s},\tilde{\mathbf{x}}}}(\mathbf{r}, \tilde{\mathbf{y}})\, d(\mathbf{r}, \tilde{\mathbf{y}}), & (\mathbf{s}, \tilde{\mathbf{x}}) \in \tilde{E}. \end{cases} \tag{69}$$

4 Martingales and martingale inequalities

In this section we construct a natural family of martingales for the extended PDMP $\tilde{X}(t)$ and prove some basic estimates.

4.1 The compensated empirical measure and test functions

We now construct a class of test functions $\tilde{h} : \tilde{E} \to \mathbb{R}$ that are analogous to (52) but lie in $\mathcal{D}(\tilde{\mathcal{A}})$. Clearly, each state $(\mathbf{s}, \tilde{\mathbf{x}}) = (\mathbf{s}, \mathbf{x}, \nu)$ defines the same empirical measure as (13):

$$\mu_\sigma(\mathbf{s}, \tilde{\mathbf{x}}) = \frac{1}{N}\sum_{i=1}^{|\mathbf{s}|} \mathbf{1}_{\{s_i=\sigma\}} \delta_{x_i} \tag{70}$$

We use both μ_σ and ν to define the *compensated empirical measure*

$$\kappa_\sigma(\mathbf{s}, \tilde{\mathbf{x}}) = \mu_\sigma + L_\sigma \delta_0 + \sum_{\rho=1}^{M} (\nu_{\sigma,\rho} - \nu_{\rho,\sigma}). \tag{71}$$

The sum accounts for all mutations from and to species σ. Finally, each $\psi \in \mathcal{C}$ may be used to construct a test function $\tilde{h} : \tilde{E} \to \mathbb{R}$ defined by

$$\tilde{h} = \langle \kappa_\sigma, \psi \rangle. \tag{72}$$

Theorem 5. *The test function $\tilde{h}$ lies in $\mathcal{D}(\tilde{\mathcal{A}})$.*

The crux of Theorem 5 is the following identity.

Lemma 3. *For each $\psi \in \mathcal{C}$ and $(\mathbf{s}, \tilde{\mathbf{x}}) \in \tilde{E}$*

$$\tilde{h}(\mathbf{r}, \tilde{\mathbf{y}}) = \tilde{h}(\mathbf{s}, \tilde{\mathbf{x}}), \quad (\mathbf{r}, \tilde{\mathbf{y}}) \in \tilde{E}^{\partial}_{\mathbf{s},\tilde{\mathbf{x}}}. \tag{73}$$

of Lemma 3. As in the discussion preceding equation(54), we decompose the neighbors $\tilde{E}^{\partial}_{\mathbf{s},\tilde{\mathbf{x}}}$ into M_- subsets corresponding to boundary events at each of the M_--species in S_-. At species l, we find that $(\mathbf{r}, \tilde{\mathbf{y}})$ is of the form $(\mathbf{r}, \mathbf{y}, \nu + \triangle\nu, (L + \triangle L)\delta_0)$ where $(\mathbf{r}, \mathbf{y})$ differs from $(\mathbf{s}, \mathbf{x})$ as described in Section 3.3, and $\triangle L$ and $\triangle\nu$ are described in equations (66) and (67).

First suppose $\sigma \neq l$. In this case, L_σ does not change at the boundary event, and

$$\tilde{h}(\mathbf{r}, \tilde{\mathbf{y}}) - \tilde{h}(\mathbf{s}, \tilde{\mathbf{x}}) = h(\mathbf{r}, \mathbf{y}) - h(\mathbf{s}, \mathbf{x}) + \sum_{\rho=1}^{M} \langle \triangle \left(\nu_{\sigma,\rho} - \nu_{\rho,\sigma} \right), \psi \rangle \,. \tag{74}$$

The second term is evaluated using equation (67).

$$\begin{aligned} &\sum_{\rho=1}^{M} \langle \triangle \left(\nu_{\sigma,\rho} - \nu_{\rho,\sigma} \right), \psi \rangle \qquad (75) \\ &\quad = \frac{1}{N} \sum_{\rho=1}^{M} \sum_{j=1}^{K^{(l)}} \psi(x_{i_j}) \left(\mathbf{1}_{s_{i_j}=\sigma} \mathbf{1}_{R^{(l)}_{s_{i_j},j}=\rho} - \mathbf{1}_{s_{i_j}=\rho} \mathbf{1}_{R^{(l)}_{s_{i_j},j}=\sigma} \right) \\ &\quad = \frac{1}{N} \sum_{j=1}^{K^{(l)}} \psi(x_{i_j}) \left(\mathbf{1}_{s_{i_j}=\sigma} - \mathbf{1}_{R^{(l)}_{s_{i_j},j}=\sigma} \right), \end{aligned}$$

after an interchange of order of summation and the use of the identity $\sum_{\rho=1}^{M} \mathbf{1}_{S=\rho} = 1$, for a fixed integer S, $1 \leq S \leq M$. By equation (54), the term above cancels $h(\mathbf{r}, \mathbf{y}) - h(\mathbf{s}, \mathbf{x})$ in equation (74).

In the event that $\sigma = l$, in addition to the above, the atom at the origin $L\delta_0$ is augmented by an atom with weight $\triangle L_\sigma = 1/N$. However, now the emprirical measure of $\mathbf{y}$ and $\mathbf{x}$ differ by an atom of size $1/N$ at the origin. The contributions of these terms cancel and the Lemma follows. □

of Theorem 5. Conditions (1) and (2) of Theorem 3 are easy to verify. In order to check condition (3), we must show that

$$\sum_{(\mathbf{r},\tilde{\mathbf{y}}) \in \tilde{E}^{\partial}_{\mathbf{s},\tilde{\mathbf{x}}}} p^{\partial}(\mathbf{r}, \tilde{\mathbf{y}}; \mathbf{s}, \tilde{\mathbf{x}}) \left(\tilde{h}(\mathbf{r}, \tilde{\mathbf{y}}) - h(\mathbf{s}, \tilde{\mathbf{x}}) \right) = 0. \tag{76}$$

But this follows from Lemma 3. (Compare equation (76) with equation (53), which does *not* hold for arbitrary $\psi \in \mathcal{C}$).

We now use some simple estimates and Lemma 2 to prove condition (4). Since $\psi \in \mathcal{C}$, we find immediately that

$$|\langle \psi, \mu_\sigma(t) \rangle| \leq \frac{1}{N} \sum_{i=1}^{|\mathbf{s}(t)|} |\psi(x_i(t))| \, \mathbf{1}_{s_i=\sigma} \leq \|\psi\|_\infty. \tag{77}$$

Similarly, we use the definition of $\nu_{\rho,\sigma}$ in equation (57) to find

$$\begin{aligned} |\langle \psi, \nu_{\rho,\sigma}(t) \rangle| &= \frac{1}{N^2} \left| \sum_{m=1}^{\infty} \sum_{l=1}^{M} \sum_{j=1}^{K^{(l)}} \mathbf{1}_{\{\tau_m^{(l)} < t\}} \mathbf{1}_{\{S_{j,m}^{(l)} = \rho\}} \mathbf{1}_{\{R_{S_{j,m},j}^{(l)} = \sigma\}} \psi(X_{j,m}^{(l)}) \right| \\ &\leq \frac{K_* \|\psi\|_\infty}{N} \sum_{m=1}^{\infty} \sum_{l=1}^{M} \mathbf{1}_{\{\tau_m^{(l)} < t\}}, \end{aligned} \tag{78}$$

where $K_* = \max_l K^{(l)}$. Then by equations (71) and (72)

$$\begin{aligned} \left| \tilde{h}(\tilde{X}(t)) \right| &= |\langle \psi, \kappa_\sigma \rangle| \leq |\langle \psi, \mu_\sigma(t) \rangle| + 2M \max_{\rho,\sigma} |\langle \psi, \nu_{\rho,\sigma}(t) \rangle| \\ &\leq \|\psi\|_\infty \left(1 + \frac{2MK_*}{N} \sum_{m=1}^{\infty} \sum_{l=1}^{M} \mathbf{1}_{\{\tau_m^{(l)} < t\}} \right). \end{aligned} \tag{79}$$

We now apply equation (59) of Lemma 2 to obtain

$$\sup_{t \geq 0} \left| \tilde{h}(\tilde{X}(t)) \right| \leq \|\psi\|_\infty (1 + 2MK_*). \tag{80}$$

Clearly, this implies Condition (4), and the proof is complete.

□

4.2 Martingales

By Dynkin's formula each $\tilde{h} \in \mathcal{D}(\tilde{\mathcal{A}})$ generates a martingale. In order to express this martingale, we re-express the total numbers and rates defined in equations (15) and (16) in terms of μ_k. (Note also that μ_k depends on N, though we suppress the superscript here).

$$F_k(t) = \langle \mu_k(t), 1 \rangle, \quad F(t) = \sum_{k=1}^{M} F_k(t), \quad F^{(0)}(t) = \sum_{k=1}^{M} w_k^{(0)} F_k(t), \quad \gamma(t) = \frac{F(t)}{F^{(0)}(t)}.$$

Theorem 6. *For each $\psi \in \mathcal{C}$ and each species σ,*

$$M_\sigma^\psi(t) := \langle \kappa_\sigma(t), \psi \rangle - \langle \kappa_\sigma(0), \psi \rangle - \int_0^t \langle \mu_\sigma(\tau), \psi' v_\sigma \rangle \, d\tau \tag{81}$$

$$+\beta \int_0^t \gamma(t) \left(K^{(0)} w_\sigma^{(0)} \langle \mu_\sigma(\tau), \psi \rangle - \sum_{k=1}^{M} w_k^{(0)} \langle \mu_k(\tau), \psi \rangle \sum_{j=1}^{K^{(0)}} \mathbf{1}_{R_{kj}^{(0)} = \sigma} \right) d\tau$$

is a martingale.

Proof. Theorem 6 follows from Dynkin's formula (39) and Theorem 5. It is only necessary to compute the action of the generator on $\tilde{h}$. We substitute equation (72) in equations (46) and (62) to obtain the drift term in (37)

$$\tilde{\mathbf{v}}\left(\tilde{h}(\mathbf{s},\tilde{\mathbf{x}})\right) = \frac{1}{N}\sum_{i=1}^{|\mathbf{s}|}\psi'(x_i)\mathbf{1}_{s_i=\sigma} = \langle \mu_\sigma, \psi' v_\sigma\rangle\,. \tag{82}$$

Next, by equations (49) and (65)

$$\lambda(\mathbf{s},\tilde{\mathbf{x}}) = \beta|\mathbf{s}| = \beta\sum_{k=1}^{M}\langle\mu_k, 1\rangle = \beta F(t). \tag{83}$$

Since ν does not change at an internal event, equations (64), and (65) and (69) imply

$$\begin{aligned}
&\int_{\tilde{E}}\left(\tilde{h}(\mathbf{r},\tilde{\mathbf{y}}) - \tilde{h}(\mathbf{s},\tilde{\mathbf{x}})\right)\tilde{Q}(d(\mathbf{r},\tilde{\mathbf{y}});\mathbf{s},\tilde{\mathbf{x}}) && (84)\\
&\quad= \int_E \left(h(\mathbf{r},\mathbf{y}) - h(\mathbf{s},\mathbf{x})\right) Q(d(\mathbf{r},\mathbf{y});\mathbf{s},\mathbf{x}) && (85)\\
&\quad= \sum_{\mathbf{r},\mathbf{y}\in E^{(0)}_{\mathbf{s},\mathbf{x}}} \left(h(\mathbf{r},\mathbf{y}) - h(\mathbf{s},\mathbf{x})\right) p^{(0)}(\mathbf{r},\mathbf{y};\mathbf{s},\mathbf{x}) && (86)\\
&\quad= \sum_{\mathbf{r},\mathbf{y}\in E^{(0)}_{\mathbf{s},\mathbf{x}}} \langle \mu_\sigma(\mathbf{r},\mathbf{x}) - \mu_\sigma(\mathbf{s},\mathbf{x}), \psi\rangle\; p^{(0)}(\mathbf{r},\mathbf{y};\mathbf{s},\mathbf{x}). && (87)
\end{aligned}$$

In the last step we have used the fact that all states $\mathbf{r},\mathbf{y}\in E^{(0)}_{\mathbf{s},\mathbf{x}}$ have $\mathbf{y}=\mathbf{x}$ since interior events mutate particles, but do not change their sizes. Finally, we recognize that the probabilities $p^{(0)}(\mathbf{r},\mathbf{y};\mathbf{s},\mathbf{x})$ are given by (7). More precisely, each state $(\mathbf{s},\mathbf{x})$ defines a law $\mathbb{P}^{\mathbf{s},\mathbf{x}}$ for $K^{(0)}$ iid integers S_j and sizes X_j. The state $\mathbf{r}$ and $\mathbf{s}$ differ in at most $K^{(0)}$ indices chosen according to this law (7). Thus, the last term in (87) may be rewritten as

$$\begin{aligned}
&\mathbb{P}\left(\sum_{j=1}^{K^{(0)}}\psi(X_j)\left(\mathbf{1}_{R^{(0)}_{S_j,j}=\sigma} - \mathbf{1}_{S_j=\sigma}\right)\right)\\
&\qquad= \frac{1}{F^{(0)}}\left(\sum_{k=1}^{M} w_k^{(0)}\,\langle\mu_k,\psi\rangle\sum_{j=1}^{K^{(0)}}\mathbf{1}_{R^{(0)}_{kj}=\sigma} - w_\sigma^{(0)}K^{(0)}\langle\mu_\sigma,\psi\rangle\right).
\end{aligned}$$

□

4.3 A regularity estimate for the case $\beta = 0$

A basic regularity estimate used to establish tightness of approximation is best understood in the special case when $\beta = 0$. Then the dynamics consists of a

deterministic flow interspersed with a random mutation at each boundary event. Despite the fact that $\mu(t)$ is stochastic, we have the following identity.

Lemma 4. *Assume $\beta = 0$. Then $M_\sigma^\psi(t) \equiv 0$.*

Proof. If $\beta = 0$, M_σ^ψ does not change at jumps as a consequence of Lemma 3. Thus, $M_\sigma^\psi(t)$ is deterministic, and a martingale, which implies

$$M_\sigma^\psi(t) = \mathbb{E}M_\sigma^\psi(t) = \mathbb{E}M_\sigma^\psi(0) = 0.$$

□

We now introduce a quantitative measure of the normalized number of particles in any interval. Define

$$m(r,t) = \sup_{a\in\mathbb{R}_+} \sum_{\sigma=1}^{M} \int_{[a,a+r)} \mu_\sigma(t)(dx). \tag{88}$$

Lemma 5. *Assume the velocity fields v_s satisfy the L^∞ bound in equation (??). Then*

$$m(r,t) \le m(3r,0)e^{2Vt/r}. \tag{89}$$

Proof. The key to the lemma is a cancellation of the mutation measure when summing over σ. We use the definition of κ in equation (71) and Lemma 4 to obtain the identity

$$\sum_{\sigma=1}^{M} \langle \psi, \mu_\sigma(t)\rangle + \psi(0)L_\sigma(t) = \sum_{\sigma=1}^{M} \langle \psi, \mu_\sigma(0)\rangle + \int_0^t \langle \mu_\sigma(\tau), v_\sigma\psi'\rangle \, d\tau. \tag{90}$$

Fix $a \in \mathbb{R}_+$. We choose a piecewise affine, positive test function

$$\psi(x) = \begin{cases} 1-(x-a)/r, & \max(a-r,0) \le x < a, \\ 1, & a \le x < a+r, \\ (a+2r-x)/r, & a+r \le x \le a+2r. \end{cases} \tag{91}$$

Since $0 \le \psi \le 1$ and $\psi = 1$ on the interval $[a, a+r)$, it is clear that

$$\int_{[a,a+r)} \mu_\sigma(t)(dx) \le \langle \psi, \mu_\sigma(t)\rangle. \tag{92}$$

Similarly, since the support of ψ is an interval of length at most $3r$, definition (88) implies

$$\sum_{\sigma=1}^{M} \langle \psi, \mu_\sigma(0)\rangle \le m(3r,0) \tag{93}$$

Finally, $|\psi'(x)| = 1/r$ on the two intervals $(\max(a-r,0), a)$ and $(a+r, a+2r)$ of length at most r and is zero elsewhere. Thus,

$$\sum_{\sigma=1}^{M} |\langle \mu_\sigma(\tau), v_\sigma\psi'\rangle| \le \frac{2V}{r} m(r,\tau). \tag{94}$$

We combine estimates (92)–(94) to obtain the inequality

$$\sum_{\sigma=1}^{M} \int_{[a,a+r)} \mu_\sigma(t)(dx) \leq m(3r,0) + \frac{2V}{r} \int_0^t m(r,\tau)\, d\tau.$$

We then take the supremum over all $a \in \mathbb{R}_+$ to obtain the estimate

$$m(r,t) \leq m(3r,0) + \frac{2V}{r} \int_0^t m(r,\tau)\, d\tau. \tag{95}$$

Gronwall's inequality now implies (89). □

5 Wellposedness of the kinetic equations

In this section, we show that the kinetic equations (17) admit mild solutions on a maximal interval of existence. In order to define mild solutions, we integrate (17) along characteristics for each species σ to obtain

$$f_\sigma(x,t) = f_\sigma(x - v_\sigma t, 0) + \int_0^t j_\sigma\left(x - v_\sigma(t-\tau)\right)\, d\tau. \tag{96}$$

Here we assume that $x \geq 0$, $t > 0$. Thus, formula (96) is well defined for all species with $v_\sigma \leq 0$, i.e. for $\sigma \in S_-, S_0$. For the species with $v_\sigma > 0$ we must use the boundary condition (24) and a priori the integral in time is defined only over the time domain $\tau \in [x/v_\sigma, t]$. However, for convenience, we extend the formula (96) to include the domain $\tau \in [0,t]$ by setting $f_\sigma(x,\tau) = j_\sigma(x,\tau) = 0$ when $x \leq 0$. It is then clear that (96) agrees with the solution obtained from the method of characteristics and the boundary condition (24).

Let X denote the space of continuous and integrable functions $f = (f_1, \ldots, f_M) : [0,\infty) \to \mathbb{R}^M$ equipped with the norm

$$\|f\| := \|f\|_{L^1} + \|f\|_{L^\infty}, \quad \|f\|_{L^1} := \sum_{\sigma=1}^{M} \|f_\sigma\|_{L^1}, \quad \|f\|_{L^\infty} := \sum_{\sigma=1}^{M} \|f_\sigma\|_{L^\infty}. \tag{97}$$

It is easy to check that X is a Banach space. We also denote

$$F_\sigma = \int_0^\infty f_\sigma(x)\, dx, \quad F = \sum_{\sigma=1}^{M} F_\sigma. \tag{98}$$

We say that $f \in X$ is *positive* if $f_\sigma(x) \geq 0$ for each σ and each $x \geq 0$. When f is positive, $F = \|f\|_{L^1}$.

Definition 3. *Assume $T > 0$ is given. A map $f \in C([0,T]; X)$ is a mild solution to (17) if (96) holds for $x \in [0,\infty)$ and $t \geq 0$. We say that f is a positive mild solution if $f(t)$ is positive for each $t \in [0,T]$.*

Theorem 7. *Assume given positive $f_0 \in X$. There exists a (possibly infinite) time $T_* > 0$ and a unique map $f \in C([0, T_*); X)$ with $f(0) = f_0$ such that f is a positive, mild solution to (17) on each interval $[0, T]$ with $0 < T < T_*$.*

Further, $\lim_{t \to T_*} f(t) = 0$ *if* $T_* < \infty$.

Theorem 7 is proved via the following Lemmas. We first show that the flux j, defined in (21), is a locally Lipschitz map. This allows us to obtain local existence of positive mild solutions by Picard's method. We then extend the solutions to a maximal interval of existence by utilizing a more careful estimate of the flux.

Let $B_r(f_0) \subset X$ denote the ball of radius $r > 0$ centered at $f_0 \in X$. As in (98) we denote

$$F_0 = \sum_{\sigma=1}^{M} \int_0^\infty f_{0,\sigma}(x).$$

We adopt the following convention in the proof. The letter C denotes a universal, positive, finite constant depending only on the parameters of the model such as the number of species M, the constant velocities v_σ, the number of mutations $K^{(l)}$ and $K^{(0)}$, the mutation matrices $R^{(l)}$ and $R^{(0)}$, the weights $w^{(l)}$ and $w^{(0)}$. It does not depend on f_0.

Lemma 6 (Uniform bounds)**.** *Assume $f_0 \in X$ is positive and non-zero. There exists $r > 0$, depending only on f_0, such that for each $f \in B_r(f_0)$.*

$$\|j(f)\| \leq C \left(\beta + \frac{\|f_0\|}{F_0} \right) \|f\|. \tag{99}$$

Proof. Recall that the flux $j(f)$ is defined by equations (21)–(22). We will estimate each term in this expression in turn.

We first estimate $\dot{L}$. We find from (20) that for every $l \in S_-$

$$|\dot{L}_l| \leq |v_l||f_l(0)| \leq \left(\max_\sigma |v_\sigma| \right) \|f_l\|_{L^\infty} \leq C\|f_0\|. \tag{100}$$

In order to estimate the weights $W_k^{(l)}$ defined by (16), we first establish a lower bound on the denominator $\sum_{p=1}^M w_p^{(l)} F_p$ for each $f \in B_r(f_0)$. Let

$$\underline{w} = \min_{\sigma,l} w_\sigma^{(l)}, \qquad \overline{w} = \max_{\sigma,l} w_\sigma^{(l)}.$$

We then have

$$\begin{aligned} \sum_{p=1}^M w_p^{(l)} F_p &= \sum_{p=1}^M w_p^{(l)} (F_p \pm F_{0,p}) \geq \underline{w} F_0 - \sum_{p=1}^M w_p^{(l)} |F_p - F_{0,p}| \\ &\geq \underline{w} F_0 - \sum_{p=1}^M w_p^{(l)} \|f_p - f_{0,p}\|_{L^1} \geq \underline{w} F_0 - \overline{w} \|f - f_0\| \geq \frac{1}{2} \underline{w} F_0, \end{aligned} \tag{101}$$

provided the radius r satisfies

$$r < \frac{w}{2\overline{w}} F_0. \tag{102}$$

We assume that r is chosen as above. It then follows from (22) and (23) that each entry in the matrix $A^{(l)}$ is bounded above by

$$|A_{\sigma k}| \leq \frac{C}{F_0}. \tag{103}$$

Thus, the operator norm $\|A^{(l)}\|_o$ of the matrix $A^{(l)}$ satisfies the estimate[3]

$$\|A^{(l)}\|_o \leq \frac{C}{F_0}. \tag{104}$$

We combine (104) with (100) to see that the flux due to boundary events is bounded by

$$\| \left(\sum_{l=1}^{M_-} A^{(l)} \dot{L}_l \right) f \| \leq C \frac{\|f_0\|^2}{F_0}. \tag{105}$$

The estimates for the interior events are simpler. We use the definition of γ in (16) and the lower bound (101) to obtain the estimate

$$0 \leq \gamma \leq C \frac{F}{F_0} \leq C, \quad f \in B_r(f_0). \tag{106}$$

It follows from the definition of $A^{(0)}$ in (22)–(23) that $\|A^{(0)}\|_o \leq C$. Thus, the flux from interior events is bounded by

$$\|\beta\gamma A^{(0)} f\| \leq C\beta \|f_0\|. \tag{107}$$

We combine estimates (105) and (107) to complete the proof. □

Lemma 7 (Lipschitz estimate)**.** *Let f_0 and r be as in Lemma 6. Then for every $f, g \in B_f(f_0)$*

$$\|j(f) - j(g)\| \leq C \left(\beta + \frac{\|f_0\|}{F_0} \right) \left(1 + \frac{\|f_0\|}{F_0} \right) \|f - g\|. \tag{108}$$

Proof. We use the expression (21) to obtain the inequality

$$\begin{aligned} &\|j(f) - j(g)\| \\ &\quad \leq \sum_{l=1}^{M_-} \|A^{(l)}(f)\dot{L}_l(f)f - A^{(l)}(g)\dot{L}_l(g)g\| + \beta\|\gamma(f)A^{(0)}f - \gamma(g)A^{(0)}g\|. \end{aligned} \tag{109}$$

[3] $\|A\|_o := \sup_{|v|=1} |Av|$, with $v \in \mathbb{R}^M$, $|v|^2 = \sum_{j=1}^M v_j^2$. Since M is finite any norm may be chosen.

Let l be fixed. It is clear that

$$|\dot{L}_l(f) - \dot{L}_l(g)| = |v_l||f(0) - g(0)| \leq C\|f - g\|. \tag{110}$$

For each k, the difference $\left|W_k^{(l)}(f) - W_k^{(l)}(g)\right|$ is estimated as follows. Let $G_p = \int_0^\infty g_p(x)\,dx$ and $G = \sum_{p=1}^M G_p$. Then

$$\begin{aligned}\left|\frac{w_k^{(l)}}{\sum_{p=1}^M w_p^{(l)} F_p} - \frac{w_k^{(l)}}{\sum_{p=1}^M w_p^{(l)} G_p}\right| &= \frac{w_k^{(l)}\left|\sum_{p=1}^M w_p^{(l)}(F_p - G_p)\right|}{\left|\sum_{p=1}^M w_p^{(l)} F_p\right|\left|\sum_{p=1}^M w_p^{(l)} G_p\right|} \\ &\leq \frac{C}{F_0^2}\sum_{p=1}^M \|f_p - g_p\|_{L^1} \leq \frac{C}{F_0^2}\|f - g\|,\end{aligned} \tag{111}$$

using (101). It then follows from (22) and (23) that each term in the matrix $A^{(l)}(f) - A^{(l)}(g)$ satisfies an estimate as above, so that

$$\|A^{(l)}(f) - A^{(l)}(g)\|_o \leq \frac{C}{F_0^2}\|f - g\|. \tag{112}$$

Finally, we use the estimates (100), (104), (110) and (111) to obtain the Lipschitz bound:

$$\begin{aligned}\|A^{(l)}(f)\dot{L}_l(f)f - A^{(l)}(g)\dot{L}_l(g)g\| &\leq \|A^{(l)}(f) - A^{(l)}(g)\|_o|\dot{L}_l(f)|\|f\| \\ &+\|A^{(l)}(g)\|_0|\dot{L}_l(f) - \dot{L}_l(g)|\|f\| + \|A^{(l)}(g)\|_o|L_l(g)|\|f - g\| \\ &\leq C\left(\frac{\|f_0\|^2}{F_0^2} + \frac{\|f_0\|}{F_0}\right)\|f - g\|.\end{aligned}$$

A calculation similar to (106) yields the estimate

$$\|\gamma(f) - \gamma(g)\| \leq \frac{C}{F_0}|F - G| \leq \frac{C}{F_0}\|f - g\|. \tag{113}$$

Thus, we find (also using the fact that $A^{(0)}$ is a constant)

$$\beta\|\gamma(f)A^{(0)}f - \gamma(g)A^{(0)}g\| \leq C\beta\left(1 + \frac{\|f_0\|}{F_0}\right)\|f - g\|. \tag{114}$$

□

Lemma 8 (Local existence)**.** *Assume $f_0 \in X$ is positive and non-zero. There exists a time $T_0 > 0$ and a map $f \in C([0,T];X)$ such that f is the unique mild solution to (17) on the time interval $[0,T]$ that satisfies the initial condition $f(0) = f_0$.*

Further, $f(t)$ is positive for each $t \in [0,T]$.

Proof. Let $r(f_0)$ be chosen as in Lemma 6. It then follows from Lemma 7 that the flux $j(f)$ is locally Lipschitz. The existence of a unique mild solution now follows by a standard application of the contraction mapping theorem.

The fact that the solution preserves positivity is seen as follows. We note that the loss term in (19), may be rewritten as $j_\sigma^-(x,t) = \alpha_\sigma(t) f_\sigma(x,t)$ where

$$\alpha_\sigma(t) = \sum_{l=1}^{M_-} \dot{L}_l K^{(l)} W_\sigma^{(l)}(t) + \beta\gamma(t) K^{(0)} w_\sigma^{(0)}. \tag{115}$$

We now rewrite the kinetic equation (17) in the form

$$\partial_t f_\sigma + v_\sigma \partial_x f_\sigma + \alpha_\sigma(t) f_\sigma = j_\sigma^+, \tag{116}$$

and observe that integration along characteristics yields

$$f_\sigma(x,t) = e^{-\int_0^t \alpha(s)\,ds} f_\sigma(x - v_\sigma t, 0) + \int_0^t e^{-\int_\tau^t \alpha(s)\,ds} j_\sigma^+ \left(x - v_\sigma(t-\tau)\right)\, d\tau, \tag{117}$$

which clearly preserves positivity. □

Lemma 9 (Maximal existence)**.** *Let $f \in C([0,T];X)$ be a positive, mild solution. Then*

$$F(t) + \sum_{l=1}^{M_-} L_l(t) = F(0), \quad L_l(t) := |v_l| \int_0^t f_l(0,s)\,ds, \quad t \in [0,T]. \tag{118}$$

There also exists a universal constant $C > 0$ such that

$$\|f(t)\|_{L^\infty} \le \|f(0)\|_{L^\infty} e^{C\beta t} \left(\frac{F(0)}{F(t)}\right)^C, \quad t \in [0,T]. \tag{119}$$

Equation (118) expresses conservation of the total number density of the system. The bound (119) degenerates if and only $F(t) \to 0$, i.e. if and only if $F_p(t) \to 0$ for each $p = 1, \dots, M$, as t approaches a critical time, say T_*. It is well-known that continuous mild solutions on an interval $[0,T]$ can be uniquely continued onto a maximal interval of existence $[0,T_*)$, such that $\lim_{t\to T_*} \|f(t)\|_X = +\infty$. Thus, the above estimates suffice to complete the proof of Theorem 7.

Proof. 1. The conservation of number for the kinetic equations is a consequence of the switching rules for the particle system. We use the identity (11), and the definition of the fluxes in equations (18) and (19) to obtain the identity

$$\sum_{\sigma=1}^{M} j_\sigma = 0. \tag{120}$$

It follows from (17) and (120) that

$$\sum_{\sigma=1}^{M} \partial_t f_\sigma + v_\sigma \partial_x f_\sigma = 0. \tag{121}$$

We integrate over $x \in [0, \infty)$ to obtain the identity

$$\frac{dF}{dt} = \sum_{\sigma=1}^{M} v_\sigma f(0,t) = -\sum_{l=1}^{M_-} \dot{L}_l. \tag{122}$$

The integral form of this identity is (118).

2. In order to prove (119) we refine the quadratic estimate of the flux in Lemma 6. We replace the lower estimate (101) by the estimate

$$\sum_{p=1}^{M} w_p^{(l)} F_p(t) \geq \underline{w} \sum_{p=1}^{M} F_p(t) = \underline{w} F(t). \tag{123}$$

We then find that

$$W_k^{(l)}(t) = \frac{w_k^{(l)}}{\sum_{p=1}^{M} w_p^{(l)} F_p(t)} \leq \frac{\underline{w}}{\overline{w}} \frac{1}{F(t)} = \frac{C}{F(t)}. \tag{124}$$

We then combine equations (22) and (23) with inequality (124) to obtain the pointwise estimate

$$\|A^{(l)}(t)\|_o \leq \frac{C}{F(t)}, \quad t \in [0,T]. \tag{125}$$

Consequently, the flux due to boundary events satisfies the L^∞ estimate

$$\begin{aligned} \|\sum_{l=1}^{M_-} A^{(l)} \dot{L}_l f(t)\|_\infty &\leq \sum_{l=1}^{M_-} \|A^{(l)}(t)\|_o \dot{L}_l \|f(t)\|_\infty \\ &\leq C\|f(t)\|_\infty \frac{\sum_{l=1}^{M_-} \dot{L}_l}{F(t)} \overset{(122)}{=} -C\|f(t)\|_\infty \frac{\dot{F}}{F}. \end{aligned} \tag{126}$$

The flux due to interior events is controlled in a similar manner. As in (106) we find $\gamma(t) \leq C$, $t \in [0,T]$. Since $\|A^{(0)}\|_o \leq C$, we find

$$\|\beta\gamma(t) A^{(0)} f(t)\|_\infty \leq C\beta \|f(t)\|_\infty. \tag{127}$$

We combine (126) and (127) to obtain

$$\|j(t)\|_\infty \leq C\|f(t)\|_\infty \frac{d}{dt}\left(\log \frac{F(0)}{F(t)} + \beta t\right). \tag{128}$$

We now substitute these L^∞ estimates in the solution formula (96) to obtain

$$\begin{aligned} \|f(t)\|_\infty = \sum_{\sigma=1}^{M} \|f_\sigma(\cdot,t)\|_\infty &\leq \sum_{\sigma=1}^{M} \|f_\sigma(\cdot,0)\|_\infty + \int_0^t \|j_\sigma(\cdot,\tau)\|_\infty \, d\tau \\ &\overset{(128)}{\leq} \|f(0)\|_\infty + C\int_0^t \|f(\tau)\|_\infty \frac{d}{d\tau}\left(\log \frac{F(0)}{F(\tau)} + \beta\tau\right) d\tau. \end{aligned} \tag{129}$$

An application of Gronwall's lemma, yields (119).

□

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